A knot theory for eight year olds?

In the first of a series of articles, Dan Ghica describes working on complex algebra with very young children.

Last year, I experimented with teaching seven- and eight-year-olds about monoidal categories using a concrete model of cardboard rectangles and bits of string. The experiment was a success, with children becoming able to represent formally a model, or to construct a model from a formal specification. They also developed an understanding of some key concepts, such as coherence. The physical manipulation of cardboard and strings was fun and relating it to notation and equations was surprising to the children and, I think, interesting.

This year I am reprising the experiment, with some changes. The first thing I was not too happy with last year, was the overhead of constructing models of cardboard and string. Too many things can go wrong, from poking holes too close to the margins so that they tear, to making the wires too long or too short. Not to mention that some children did not know how to tie a knot. The motivation of building structures of cardboard and strings was also unclear, I did not even have a name for these structures. So, this year we are only looking at knots. They are simpler to build since all you need is the strings. No more cardboard and hole punchers. They are also something children already know. And if they do not know how to tie them it is okay, because that is what we are learning. The second change I am making is slowing down the pace and making the process more exploratory. I will not try to teach them anything in particular, but we will aim to develop together an algebra for knots.

The first meeting of our club started with a general discussion about what is mathematics. Seven- and eight-year-olds are taught a very narrow view of the subject, which they tend to identify with arithmetic. “Tricky problems” was another common view of what mathematics is about. When I asked whether a triangle is something that mathematics deals with the kids started to question their own understanding of what mathematics is really about.

I proposed that mathematics is the general study of patterns and this was immediately accepted. Then we talked a while about the need in mathematics to represent concepts and I introduced the concept of “notation”, illustrated with a potted history of numerals, notations for the numbers. We spent some time on hieroglyphic numerals (see figure 1), a topic which turned out to be very enjoyable in its own right. I wanted children to discover the inconveniences of this system compared with our Indo-Arabic system and they indeed came to that conclusion quite soon.

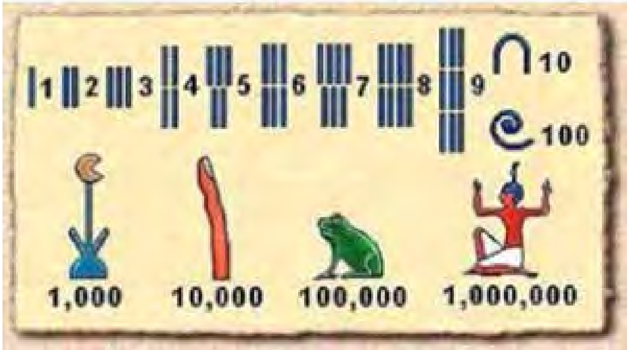

Figure 1: Hieroglyphic numerals.

The next step was to introduce knots. The fact that mathematics would have something to say about knots struck many children as implausible but even the most sceptical changed their tune immediately when reminded that “knots are some kind of patterns we make with strings” and “mathematics is about patterns”. This abrupt change of mind and wholehearted acceptance of a previously rejected thesis makes working with these children a great pleasure.

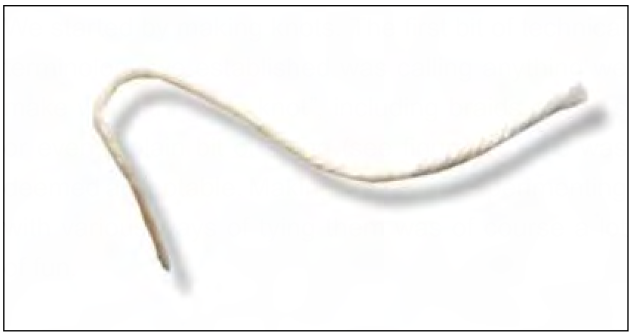

Figure 2: This is also a knot.

The second step was representing knots by drawing them. The challenge was to draw a half-knot (see figure 3):

Figure 3: A half-knot

The children under-estimated the difficulty of the task, producing something like this (see figure 4):

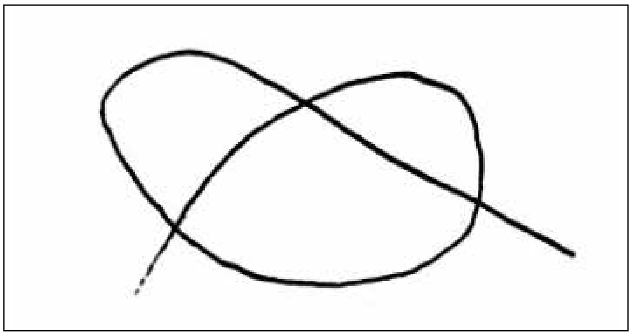

Figure 4: Drawing of a half-knot.

This drawing of course makes line crossing ambiguous. Which line is above, and which line is below? I suggested leaving small gaps to indicate line crossings (see figure 5):

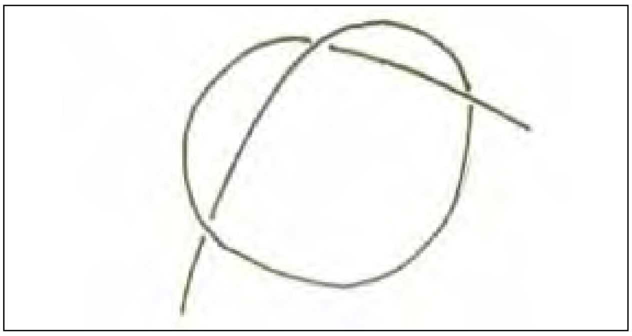

Figure 5: Second drawing of half-knot.

The principle was readily understood but the execution was problematic. Drawing the knot correctly proved a real challenge, with more than half the children failing to draw it correctly.

The upshot of all this struggle was a consensus that we really need a better way to write down knots, a "notation" (knotation?). The groundwork has been laid for creating algebra for knots which you will read about in the next issue.

Dan Ghica is a reader in the semantics of programming languages at the University of Birmingham.

This article first appeared as a blog in 2015. The original blog is no longer available. The club was held at Water Mill Primary School in Birmingham.

Conference 2018.

A knot theory for eight-year-olds: Part 2

This is the second in the series of pieces by Dan Ghica in which he describes how he introduced young learners to complex algebra.

You will have read in the last issue that these articles reflect on the process of running a mathematics club for 8-year-old children, in which we are reinventing (an) algebraic knot theory. We are not trying to reinvent existing knot theory, just to make a journey of intellectual and mathematical discovery. The previous article described how I gently shocked the children by showing them that mathematics is not only about numbers, but about patterns in general, such as knots. Being eight-year-old children, they are very theatrical in expressing emotion, "What! Maths about knots?" I loved the wild face expressions and hand gestures. We spent a lot of our time making and drawing knots and concluded that drawing knots is quite hard. We needed a better notation.

A notation for knots: the "knotation"

One preliminary observation that I made, which resonated with the children, was that there are a lot of numbers out there but we only need 10 symbols to represent all of them: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Of course, we could only use 0 and 1, or just 1, but let us not get distracted. It is about having a nice and convenient notation not just any notation. So, I drew a couple of knots on the whiteboard and I encouraged the children to try to identify the "smallest pieces the knot is made of", where by a "piece" I meant this: We draw a small circle on the whiteboard and the "piece" is the bit of knot we see inside the circle. If our circles are small enough it is easy to see that the pieces inside the circles are quite similar. So here is an overhand knot with some parts of it circled (see figure 1).

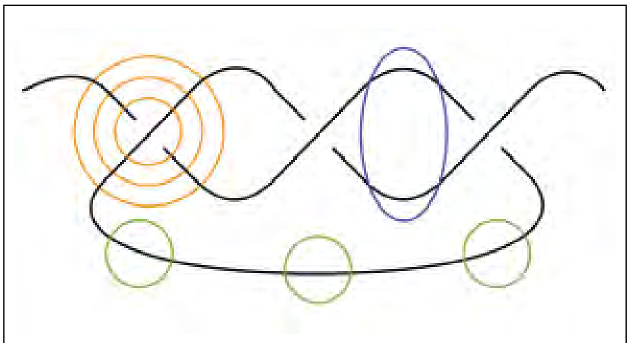

Figure 1: An overhand knot.

We noticed that even if we 'zoom in' on a knot piece, as in the case of the red circles, the interesting knot piece 'looks the same' (see figure 2):

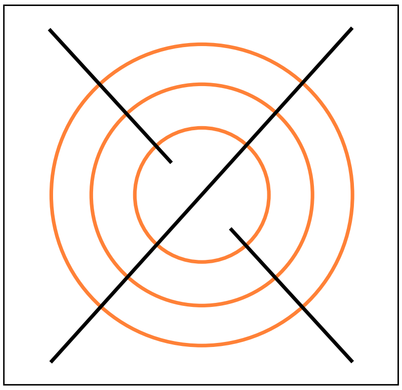

Figure 2: A knot piece.

We noticed that if a knot piece is too large, as in the case of the blue circle, it can be decomposed into smaller basic knot pieces. We also noticed that many knot pieces, which we select here and there, look quite similar, as in the case of the green circles. I do not think there is a clear methodology one can use in inventing a good notation for a concept or set of concepts. It is a fun and creative endeavour. It is a design exercise in which many criteria must be balanced.

On the one hand we have expressiveness, which means that we can use the notation to represent a large range of concepts. But we also have elegance, which is more subjective but not entirely arbitrary. We want a small, but not too small, set of notations. We want it to be also concise. We want it to be pretty. So, we explored a while and in the end, with some careful guidance, we narrowed in on these constants (see figure 3):

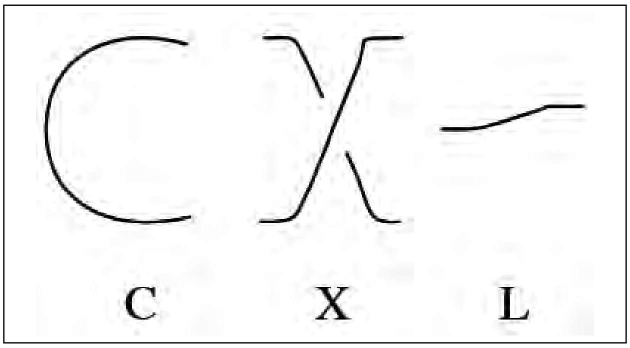

Figure 3: A "knotation".

The names C and X were suggested because of the shape of the knot piece they represent. L was short for “line”. These shapes can be flipped over and result in 3 other constants (see figure 4):

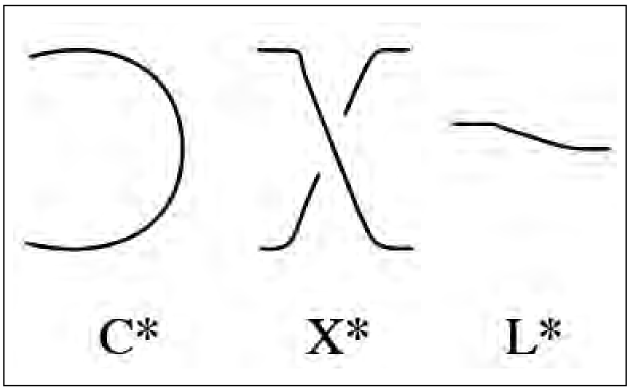

Figure 4: 3 further constants.

Here there was some disagreement between me and the children. They were happy with the flipping operation and they were happy with X* but they did not like C* because it looked more like a D, without the vertical bar. Some of them insisted that we call them D and X*. L* was a non-issue because it was just like L. I put my foot down and we settled on C, X, L and the * operator. This exercise also presented us with our first equation: L = L*. I did not insist on equations as they will become more important later. The final part of the hour was dedicated to developing notations for assembling knots out of these constants. There are two ways to compose knot pieces, horizontally and vertically. Here is how the overhand knot can be split into basic components (see figure 5):

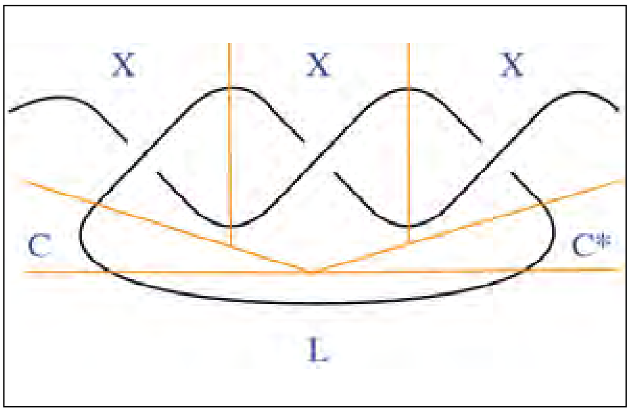

Figure 5: Splitting an overhand knot.

How do we put them together? The bits at the top can be written as X·X·X. The bit below seemed like C·C*. And at the bottom we have L. Because the bit at the bottom is quite long many children wrote it as L·L or L·L·L but they quickly realised our second equation, L = L·L. This observation led to an unfortunate turn of events in which the children decided that L is like 0 because added with itself stays the same, and therefore changed it from L to 0. I decided that in the next session I would change it to the correct one even if I need to exercise my executive power.

Then we ran out of time without designing a good notation for vertical assembly. Because we did not have vertical assembly we could not really test our notation. The way C, C* and L are composed is not entirely clear because vertical and horizontal composition interact. But, we had made a start. I wanted the children to move away from how they are used to studying mathematics, as a stipulated set of rules in which answers are either correct or incorrect. I did not want them to simply execute the algorithm, but to invent the programming language in which it is written. Coming up with stuff that does not work is okay, as long as we notice it does not work and we improve on it.

What did we accomplish? Quite a lot. In one hour, a dozen eight-year-old children rediscovered some of the basic combinators used in monoidal categories, such as the braid, or such as duality, unit and co-unit as used in compact closed categories. We also rediscovered composition and its identity, and some of its equations. We used none of the established names and notations, but we discovered the concepts. We were on our way to inventing a categorical model of knots and braids. The next article will describe how we figured out the monoidal tensor and moved on to the really exciting bit, coherence equations.

Dan Ghica is a reader in the semantics of programming languages at the University of Birmingham.

This article first appeared as a blog in 2015. The original blog is no longer available. The club was held at Water Mill Primary School in Birmingham.

A knot theory for eight-year-olds: Part 3

The third article in which Dan Ghica describes how he introduced young learners to higher algebra.

Unlike the previous two pieces, this is a description of some teaching that preceded the mathematics club meeting. I felt the need to tighten some loose ends in our emerging knot-theory notation. The first two meetings (see parts 1 and 2 in MT264 and MT265) were about exploring knots and ways to describe them using notations and we made some impressive strides. But at this stage, I felt the need to steer the children in the right direction. So, I decided:

- We are going to use *I* rather than 0 (zero) to denote the identity of composition. Even though 0 is a neutral element for addition, as correctly spotted by the children, using it as the neutral element for composition is not idiomatic. *I* or 1 (one) are more common, for various reason I will not get into here.
- Using a fraction-like notation for "parallel" (tensorial) composition is also quite un-idiomatic, although it is intuitive. I shall introduce the standard $\otimes$ notation instead.
- We will stick with a generic duality operation _* so that our "unit" and "co-unit" are C and C* rather than C and D as some of the children insisted on.

These are of course small matters which we could put up with, at the expense of becoming incomprehensible to the rest of the world. The more important thing, for me, is that the interplay between sequential (functional) composition and parallel (tensorial) composition is not very easy to get right. Look at our overhand knot (see figure 1):

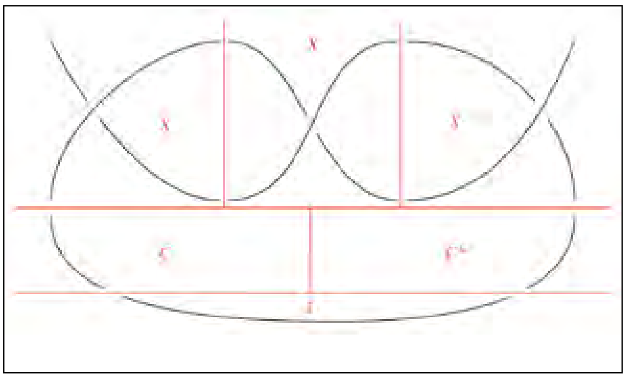

Figure 1: An overhand knot.

The individual components are well identified but it is unclear how to put them together. There is a sequential composition at the top ($X \circ X \circ X$) but how to connect that with the C, I and C^* underneath is not clear from the decomposition. X interacts with C and C with I but not obviously sequentially or in parallel.

The way out is to introduce the concept of 'types'. We can give knots a type $(m, n) \in N^2$. The left projection represents how many "loose ends" stick out to the left and the right projection how many to the right. Types tell us what can correctly compose with what. So, this is a (4, 6)-knot:

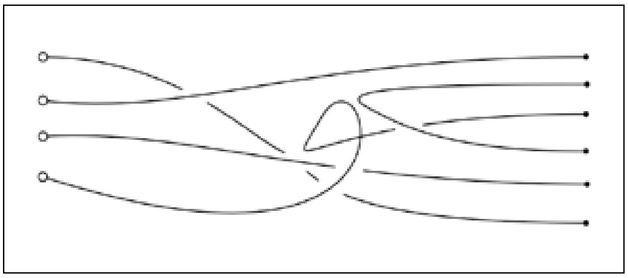

Figure 2: A (4, 6)-knot.

Note the implicit restriction: no loose ends are allowed to stick out from the top or from the bottom, because if a bit of string goes down or up we can always bend it left or right then extend it until it lines up nicely with the other loose ends. I hoped this topological invariance of knots will be intuitively clear to my young learners.

So, the types of our basic knot parts are (see fig. 3):

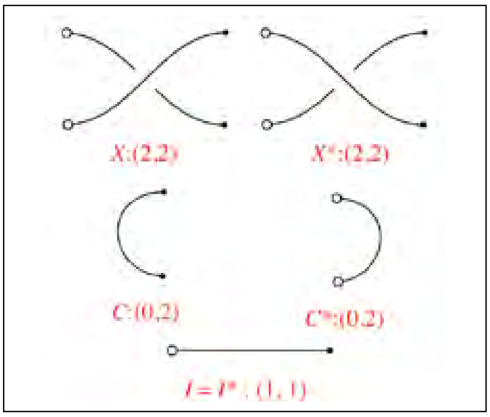

Figure 3: The basic knot parts.

Now we can say that we only allow the composition $K \circ K'$ for knots K: (m, n) and K': (m', n') if $n = m'$, resulting in a knot $K \circ K'$: (m, n'). Here are two composable knots (see fig. 4):

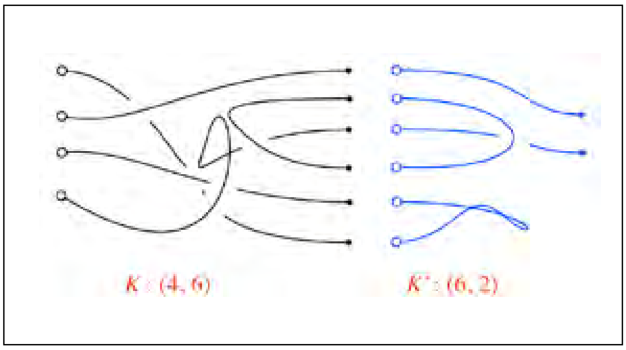

Figure 4: Two composable knots.

And this is their composition:

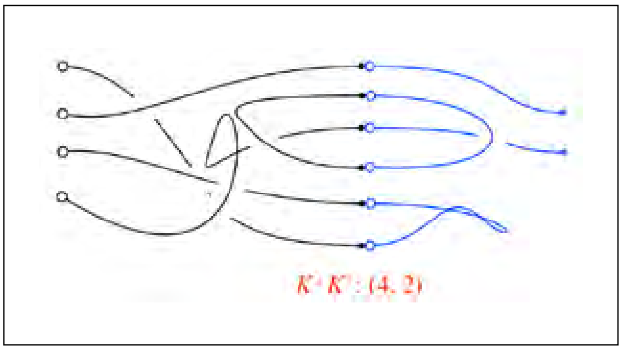

Figure 5: Knot composition.

Note that because these are strings the loose ends do not need to be geometrically aligned. It is enough to have the right number of loose ends to be able to glue them together. And this is their actual composition (see fig. 6):

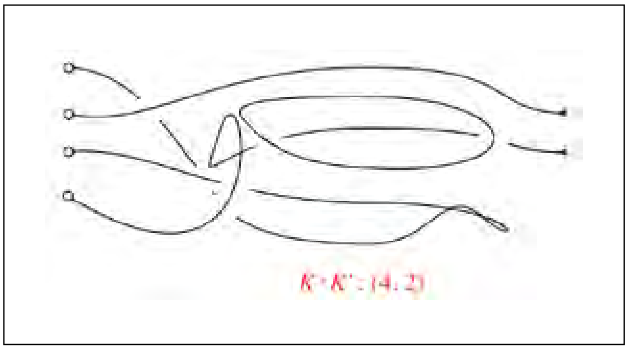

Figure 6: The actual composition.

Any two knots K: (m, n) and K': (m', n') can be composed in parallel in a knot $K \otimes K'$: $(m + m', n + n')$. The parallel composition of the two knots above is (see fig. 7):

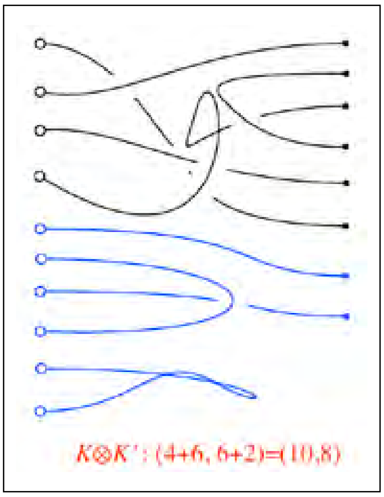

Figure 7: Composition of knots 1 and 2.

So now everything is in place to describe our overhand knot algebraically (see fig. 8):

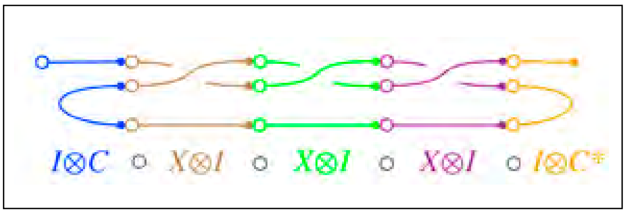

Figure 8: The algebraic notation.

The overhand knot is therefore described by the formula

$(I \otimes C) \circ (X \otimes I) \circ (X \otimes I) \circ (X \otimes I) \circ (I \otimes C^*)$.

As a simple notational improvement we can introduce -•-, a 'scalar' product, shortcut for repeated sequential composition, leading to the more concise

$(I \otimes C) \circ (3 \bullet (X \otimes I)) \circ (I \otimes C^*)$.

The identity string I, is needed, not just on the bottom, but also to the left and to the right, in order to make the loose ends match properly in sequential composition. This kind of padding with identities will be a useful trick that will allow us to define a lot of knots compositionally. Next issue, I return to our young learners.

Dan Ghica is a reader in the semantics of programming languages at the University of Birmingham.

This article first appeared as a blog in 2015. The club was held at Water Mill Primary School, Birmingham.

Inventing an algebraic knot theory for eight-year-olds: Part 4

The fourth in the series of articles in which Dan Ghica describes how he introduced young learners to higher algebra.

Here is a quotation that captures the ethos against which our mathematics club militates:

> We take other men's [*sic*] knowledge and opinions upon trust; which is an idle and superficial learning. We must make them our own. We are just like a man who, needing fire, went to a neighbor's house to fetch it, and finding a very good one there, sat down to warm himself without remembering to carry any back home. What good does it do us to have our belly full of meat if it is not digested, if it is not transformed into us, if it does not nourish and support us? (Michel de Montaigne, *Of pedantry*).

This is how mathematics is often delivered to students, as knowledge based upon trust. And it is knowledge of a particularly mystical variety, where magical symbols come together in incomprehensible ways. This is the myth I would like to dispel. This is the knowledge I would like my learners to digest. Sometimes algebra can be nothing more than a way to write down knots and their properties.

I have described in previous articles how we developed a notation for knot theory. One unexpected hiccup was that my students had not seen parentheses before, but we took that in our stride. They also had not used variables before, so most of the examples were concrete. But the little use of variables we made was not confusing. I even introduced some formal mathematics terminology. Firstly, I asked them to give names to the composition operation, and they came up with some reasonable suggestions such as "gluing", "linking", or "joining". When I told them the more common word was "composition", they liked it. They said it was, "like music", where you, "compose" a song from notes. They also had good suggestions for the tensor, things such as "stacking" or "building", but they did not much like the more formal "tensor". I do not like it much myself, I confess.

For the first 20 minutes we just tidied up notation. The rest of the time, 30 minutes or so, we worked out examples of going from a drawn knot to a formula. It is not easy, in fact it is quite a challenge, so they often got things wrong. One pervasive error was getting confused by repeated tensoring ("stacking") of tangles, K⨂K'⨂K", which often was written as K⨂K'K", an interesting error.

In the next session, we finally reached equations. We had already seen the equation I*= I, which was noticed while developing the notation. We started with similar coherence equations, which means that two notations denote the same knot. Things like associativity of composition or tensor are typical coherence equations. But, because the children have no appreciation of parentheses and operator precedence associativity is perhaps not the best example. I think functionality of the tensor is much more interesting. Think about these two examples:

1. Draw the tangle for (C◦C*)⨂(C◦C*).
2. Draw the tangle for (C⨂C)◦(C*⨂C*).

In both cases the result is a couple of loops (see figure 1). The difference is in how we put the loops together and not in what the result is:

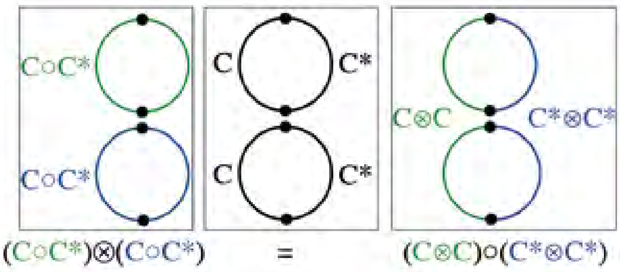

Figure 1: Two loops.

Functoriality means that no matter what tangles H1, H2, H3, H4 (for some reason the children chose *H* as the standard variable for knots) the following two tangles, if they can be constructed, are always equal:

(H1◦H2)⨂(H3◦H4)=(H1⨂H3)◦(H2⨂H4).

The unit and the co-unit also suggest compact-closed-like coherence axioms, which have a more topological flavour. Try this knot: (I⨂C)◦(C*⨂I). It looks like this (see figure 2):

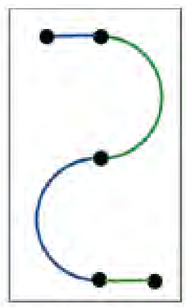

Figure 2: The knot *(I⨂C)◦(C*⨂I)*.

But this is only a wire with some irrelevant bends. We have the equation

$$(I\otimes C)\circ(C^*\otimes I) = I$$

There is another similar equation, where the shape is like an 'S' rather than like a 'Z'. Can you define it?

The trove of equations is deep for this algebraic theory of knots and tangles and there is plenty of scope for children to discover equations once they understand what they represent. In addition to compact-closed we can also find equations from other well-studied mathematical structures, such as *traced*, *braided* and *closed-monoidal* categories (see the note at end of the article). Most importantly, I think the point is not to just give such equations to the students in order for them to develop skill in calculating knot equivalences using equational reasoning, so I will not do that. That is the kind of mystical mathematics I want to avoid. What is important to me, are two points.

The first one is understanding the idea of equational reasoning. The same structure can be expressed in several equivalent ways, and that equations are the algebraic way to show this. The concrete tangles give us a *model-theoretic* view of the language of diagrams, in which we can make wholesale judgements as to whether two formulae represent the same tangle. But the equational, algebraic approach should show how instead we can identify a small number of key equations, which we can then use to show in a stepwise manner that two formulae denote the same diagram.

Contrasting and comparing the *model-theoretic* and the *equational* approach, the issues of *soundness* and *completeness* should pop up naturally. Is equational reasoning always consistent with the model-theoretical interpretation, or are some equations mistaken? Can all formulae describing the same tangle be actually shown to be so using the equations, or are we missing some?

Dan Ghica is a reader in the semantics of programming languages at the University of Birmingham.

This series of articles first appeared as blogs in 2015. The club was held at Water Mill Primary School, Birmingham.

Note: If you are interested in exploring these ideas see Selinger, P. (2010). *New structures for physics.* Berlin: Heidelberg pp 289-355.

Conference 2019

Inventing an algebraic knot theory for eight-year-olds: Part 5 - towards equations

The final piece in a series of articles in which Dan Ghica describes how he introduced young learners to higher algebra.

We finally reconvened our mathematics club after a mid-term school break followed by the university examinations period. I assumed, correctly, that nobody would remember much of what we had talked about before the break. At least not in detail. In addition, we had a significant number of new children joining the group, who had no idea what we have been up to.

In the previous sessions (see MT267) we discovered and invented our notation, which consisted of three combinators *(X, C, I)* and three operations: composition ("gluing"), tensoring ("stacking") and dual ("flipping") (see figure 1).

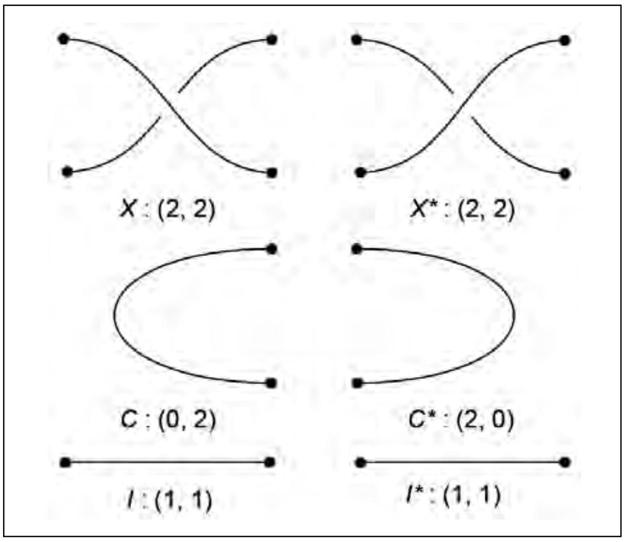

Figure 1: Combinations and operations.

This time we just took this as a given, the starting point of our development. If the previous sessions had been analytic, trying to decompose knots into basic constants and operations, this session was synthetic, starting with formulas, drawing them up and seeing what we get.

The first formula we contemplated was *C;C*,* which everyone immediately identified as a circle. The knot-theoretic name for this is "unknot", a word which caused a certain amount of amusement. One student cleverly observed, unprompted, that *(C;C*)** would have been the same thing, also an unknot, and we had a short discussion about how symmetry of a knot *K* boils down to *K=K*.* That was quite rewarding and fun and prepared the ground, somewhat serendipitously, for the next formula *C;X;C** which is drawn as below:

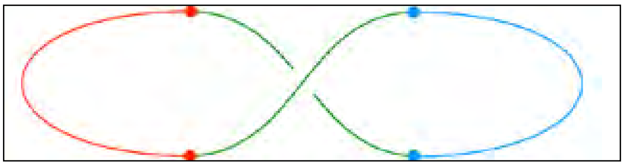

Figure 2: Another form of the unknot.

The same student pointed out that this shape is also symmetric, i.e. *C;X;C*=(C;X;C*)*.* This observation was challenged by other students and the ensuing discussion was, for me, the most interesting and important part of the session. Initially the students were in majority supportive of the validity of the equation, then someone pointed out that *X≠X** so, they reckoned, it must be that

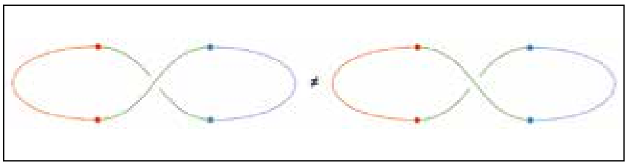

Figure 3: The symmetry of the unknot.

Indeed, the shapes are not identical. The opinion of the group swung so they all agreed that the equation is not valid and the two shapes (original and flipped version) are not equal.

But are they? What do you think? We are talking knots here, and two knots should be equal when they are topologically equal, that is that they can be deformed into each other without any tearing and gluing. So, in fact they are equal because:

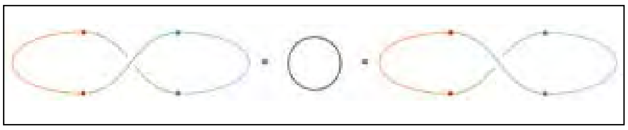

Figure 4: Symmetry from transitivity of equality.

So here we have a genuinely interesting equation, where two formulae are equal not because the shapes they correspond to are geometrically equal (that is identical) but because they are topologically equal, that is there is a smooth deformation between them. Also note that the deformation must be in three dimensions by twisting the left (or right) side of the unknot. A two-dimensional transformation would make the wire pinch which may or may not be allowable, depending on how we precisely set things up. The point was quickly grasped, and we moved on. It is a subtle point which I think can be damaged by over-explaining if the initial intuitions are in the right place, which they seemed to be.

Next we looked at a much more complicated formula that took a long time to draw correctly: $C;(I\otimes C\otimes I);(X\otimes X^*); (I\otimes C^*\otimes I);C^*$.

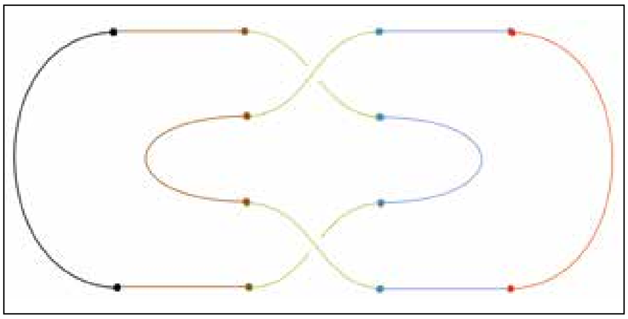

Figure 5: Two unknots, obfuscated.

As you may see, if we are drawing this as a knot, and tidy things up a bit, we actually have this:

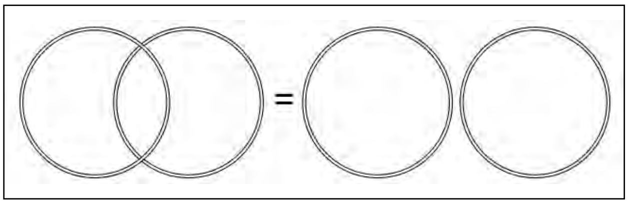

Figure 6: Two unknots, equationally.

Which is really just two unknots. They don't interlock so they can be pulled apart with no tearing and gluing,

The formula for this is the much simpler $C;C^*;C;C^*$! This point was not easy to make, and it seemed difficult for the students to appreciate it. By this time, they were quite tired and the drawing of the more complex formulation of the two unknots diagram drained them of energy and patience. They made quite a few errors and had to restart several times. Showing them a much easier way to achieve the same knot diagram almost made them angry. I hoped to use this as a motivation for the future: how to simplify a formula before we draw it.

However, for various reasons the math club did not reconvene. But by the time we reached this point the main question I had asked myself was already answered: KS1 students have the ability to not just comprehend algebraic concepts, but also to re-discover them if provided with some careful and discreet guidance. The exposure to what we called "maths without numbers" was almost shocking to them. It was a facet of mathematics that they did not imagine, and they could not have imagined extrapolating from their mathematical educational experience to date. It was a kind of mathematics that was not about what is the "right" and what is the "wrong" answer to a question, but one that was about asking questions. It was about discovering concepts and the relationships between them.

This conceptual reorientation is neatly captured by the term "congressive mathematics", coined by Eugenia Cheng. (See, *Having your Pi and eating it – talking maths with Eugenia Cheng.* https://www.cai.cam.ac.uk/news/having-your-pi-and-eating-it-talking-maths-eugenia-cheng). We mean to take nothing away from the merits of conventional ("ingressive") mathematics. Its preoccupation with correctness, teaches rigour. Its focus on problem solving, and its valuing of technical challenge builds mental strenght. But the softer approach confirmed emphatically her thesis, that congresssive mathematics is less intimidating and more engaging. In particular, students who dislike the competitive approach inherent in problem solving and the intellectual pressure of possibly being wrong seemed to benefit from the dialectical, interactive, and broadly collaborative approach.

It is difficult to say how, but following this experience I came to believe rather strongly that an injection of congressive mathematics into the curriculum could greatly broaden the appeal of mathematics as a discipline. This is true at all levels, but in particular at the early stages when students can be both discouraged and put off by the conventional, ingressive, style of teaching.

Use of Maths in Modern Life Explored in 2019-20 Gresham lectures

Students considering Maths A Level and Maths degrees may find Chris Budd's free Gresham Maths lectures useful, as they continue to focus on the applications of Maths in modern life. Next year's lectures cover the maths of computing, voting, curing cancer, and maths in art; Budd will also lecture on mathematical misconceptions and on equations that have changed the world. The lectures are live-streamed online and in a new initiative, Gresham is encouraging schools and colleges to book seats for lectures that are otherwise first-come, first-served to the public – you can get in touch on enquiries@gresham.ac.uk; you can also find them on **www.gresham.ac.uk** and sign up for updates about lectures here: **gres.hm/schools**.

Gresham College is a charitable trust that was set up in 1597 to bring free education to the public in London, and today live-streams its public lectures to reach a global audience.

Budd's most popular Maths lecture last year was on the Maths of Climate Change.

https://www.youtube.com/watch?v=w4O4jK-lZrI