Condensed Matter Field Theories

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April 18, 2018

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Preface

This stream of consciousness tries to provide a field theoretical perspective of condensed matter physics.

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Review as an Introduction

When we look around, we see stuff (many of us do, if not all). Chalk and cheese. Some of us even wonder what is common between chalk and cheese (the difference is, hopefully, clear to all!). After bit of thought we realize that both of these are "solids" – "something" that "resists shape change"; more technically stated, "something" that has a nonzero shear modulus (at zero frequency). The last statement has made the notion of "resists shape change" precise. Let us now try to make the notion of "something" clear. Take cheese – we know that it is made of "milk atoms", yet cheese *is* different from milk. More importantly, we realize that the same "milk atoms" can organize themselves into milk (liquid) or cheese (solid)! Thus, the "something" is a *phase* – a state of *many "milk atoms"* that has a distinct feature – ability to resist shape change. Moral:

> PHASES
>
> *A large number objects ("degrees of freedom") that interact with each other can organize themselves into distinct phases.*

A little reflection will show that much of human activity is focused on making different phases from some set of degrees of freedom. For example, in making "better" (phases with desired properties) materials from a collection of atoms.

A natural question to ask is this: Suppose we are given a large number of degrees of freedom and the interactions between them, what are all the distinct phases that they can organize themselves in? Condensed matter physics is the subject that attempts to answer this question – at least this is what we mean by "condensed matter physics". One might have been brought up to think that this is "a matter of detail" – we "just" have to solve the many body problem. Some of us even think that there are more

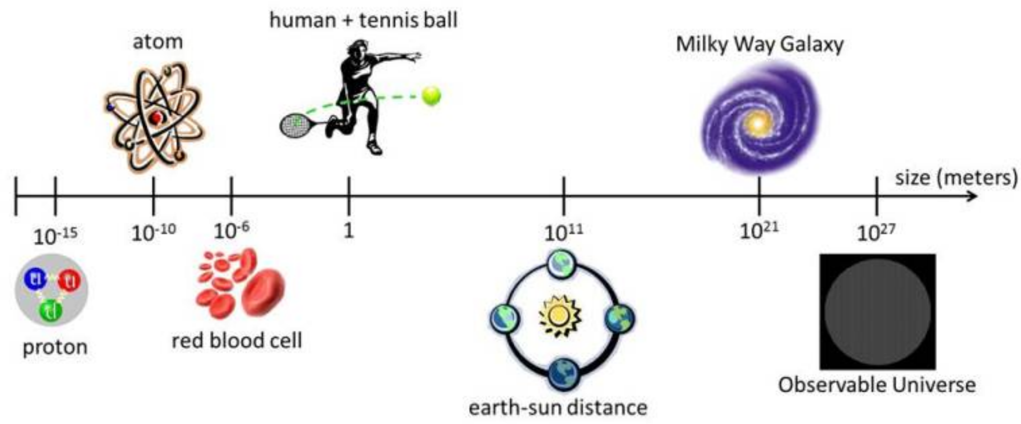

Figure 1.1: Our experience across the scales. [**?**] *fig:scales*

"fundamental" questions like – "what are the ultimate objects that make stuff up?"

Our approach is strongly influenced by what we *see*; we want to think about things that we see[1]. So, what *do* we see? See fig. 1.1. We learn a couple of things...

- At every scale, there are some "natural" degrees of freedom that interact with each other to "produce" phases at the "next bigger" scale. E. g., the liquid phase of water is made of water molecules.

- Objects that appear "fundamental" at any given scale are themselves "made of" even "more fundamental" objects of a smaller scale.

and even puzzle...

- How the "information" from the smallest scale "reach" the largest scale? Colloquially, how does the mass of the Higgs affect the viscosity of water?

The gist of all this: We are interested in what physics emerges at a certain "long wavelength" scale (infrared physics) when we know the "microscopic degrees" of freedom and their interactions at a smaller scale (ultraviolet physics).

[1] As opposed to want to see things that we think about!

THE QUESTION

The question is posed precisely, by specifying: (i) the ultraviolet scale (ii) microscopic degrees of freedom at that scale, and (iii)their interactions. Once this is done, the question to be answered is what is the infrared physics? What sort of phases emerge?

We have at hand new issues to discuss. What exactly do we mean by a microscopic degree of freedom (ultraviolet question)? And what do we really mean by a phase (infrared question)? These are indeed somewhat difficult notions, and we will aim for clarity rather than precision.

One way to introduce degrees of freedom at the ultraviolet scales is to have some "physical idea" regarding the symmetries of the system at the that scale. Suppose, we believe (a question that can only be settled by experiment) that the system has a symmetry group G on the ultraviolet scale, the microscopic degrees of freedom with be some sort of representation of the group realized on an *arena*. Arena here stands for space, time or space-time. In other words, given a collection of arena points, each point of the arena has a representation of the group G associated with it – this leads naturally to the notion of *field* particularly if the arena is a space or space-time continuum. We will also include lattices as arenas, and the resulting theories are really lattice field theories, although we will call these as field theories. Symmetries may be classified as being *arena symmetries* or *internal symmetries*. Symmetries determine the allowed terms in the Hamiltonian of the system; typically this specified "rules" for how fields at distinct arena points interact with each other. Quite interestingly, this can in principle lead a (an infinitely) large number of terms – "everything" allowed by symmetry (we also sometimes require "locality", a notion that we will not explore).

Sounds abstract mumbo-jumbo? Lets look at an example. Consider the arena to be square lattice in two dimensions, with sites are labelled by i, placed in a nice periodic box. Someone tells us that our system on the scale of the lattice has all the symmetries of the square lattice (arena symmetries) and an internal symmetry $\mathbb{Z}_2$ – i. e., Ising symmetry. Follow the rules...first find a "representation"[2] of $\mathbb{Z}_2$. A variable $\sigma \in \{-1, 1\}$ does the job – we call σ an "Ising spin". How does the symmetry act on the Ising spin? $\mathbb{Z}_2$ has two elements – I the identity and F the filp element ($F^2 = I$). The action of $\mathbb{Z}_2$ on σ is easy: $I\sigma = \sigma$ and $F\sigma = -\sigma$. Now paste an Ising spin at each arena site i (lets us say there are N sites) and call it σ_i. A field configuration is now specified by specifying the spin state at each

[2] Dont rush off to read a book on group theory.

site i, i. e., $|\psi\rangle = \{\sigma_i\}$. Good. On to the Hamiltonian...we are given that the system has all the square lattice symmetries in addition to the internal Ising symmetry. One of the simplest possible Hamiltonian is

eqn:IsingHam

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{1.1}$$

where $\langle ij \rangle$ stands for nearest neighbour bonds, where J is a coupling constant (with units of energy). Its easy to check that this system has all the required symmetries. An interesting question (whose answer we will require later) for you to answer is this: Write out a couple of more terms that respect all the symmetries of the system.

The next question: What all phases can such an Ising system realize? We will continue to work with an intuitive notion of what a phase is, and in fact, ask how do we distinguish between two different phases? Let us begin by considering the ground state of the system defined in eqn. (1.1). One possible ground state is

$$|G \uparrow\rangle = \{+1_i\}. \tag{1.2}$$

Note that the this state has all the symmetries of the system, *except* the $\mathbb{Z}_2$ symmetry! In fact, action of F on $|G \uparrow\rangle$ gives the other distinct ground state

$$F|G \uparrow\rangle = |G \downarrow\rangle = \{-1_i\} \tag{1.3}$$

We say that the state is a *broken symmetry state*. Again, all this happens at zero temperature – this system picks either $|G \uparrow\rangle$ or $|G \downarrow\rangle$ as its ground state which breaks the $\mathbb{Z}_2$ symmetry of the system.

What about at a finite temperature T? First of all how does one describe the state of the system at a finite temperature. Recall from elementary statistical mechanics, that the state is described by the *thermal density matrix* ρ

eqn:PFIsing

$$\rho = \frac{1}{Z} e^{-H/T} \tag{1.4}$$

where $Z = \mathrm{tr}\, e^{-H/T}$. We could check if the density matrix is symmetric under the $\mathbb{Z}_2$ group. In other words, you are asking if the state changes under a symmetry operation. If you naively apply an $\mathbb{Z}_2$ element to ρ, you will come away with the (wrong) conclusion that ρ is $\mathbb{Z}_2$ symmetric. What you realize is that expression for ρ is really a short from for the following. First apply an "external symmetry breaking field" B

$$H_B = H - B \sum_i \sigma_i \tag{1.5}$$

and obtain ρ_h as

$$\rho_B = \frac{1}{Z_B} e^{-H_B/T}. \tag{1.6}$$

Now first take the thermodynamic limit $N \to \infty$ and *then* switch off the external field B, i. e.,

eqn:TLimit

$$\rho = \lim_{B\to 0} \lim_{N\to\infty} \rho_B \tag{1.7}$$

and, in fact, eqn. (1.4) is a short from for this expression. In particular, you realize that the ground state will be $|G\uparrow\rangle$ is the limit $B \to 0$ is taken as $B \to 0^+$. In other words, if symmetry is indeed broken, the answer for ρ will depend on *how* the limit $B \to 0$ is taken!

What if you take the limit in the opposite order? Then, you will find that the symmetry is never broken, i. e., a finite system will not break symmetry. In other words, symmetry breaking occurs only in the *thermodynamic* limit. You should understand (review) this idea very carefully before you proceed.

What if you were at infinite temperature?[3] If you work out carefully the density matrix will trun out to be

$$\rho_\infty = \text{constant} \tag{1.8}$$

(what is the constant?),i. e., every state is equally probable. It is easy to see that the state is symmetric under $\mathbb{Z}_2$!

The discussion leads us to the conclusion. At zero temperature, the system in equilibrium *breaks symmetry*, while at infinite temperature, the system is fully symmetric. This is a good point to inject the notion of "phase". We say that broken symmetry zero temperature state describes an *ordered phase*. Roughly what this means is that if you know the state of a single spin, then you will be able to say something about the state of any another spin that is even "very far away", the spins have "ordered". At $T = \infty$ the story is quite the opposite – each spin is "doing its own thing", and the system is literally disordered, i. e., in a *disordered phase*! Note that the language is funny, if a bit confusing – ordered phase $\leftrightarrow$ broken symmetry, disordered phase $\leftrightarrow$ symmetry preserved! Finally, we recall that ordered (broken symmetry) phases can be characterized by an *order parameter*, an quantity that is manifestly *not invariant* under the symmetry group. In our Ising example, the magnetization defined by

eqn:OP

$$M = \frac{1}{N} \sum_i \sigma_i \tag{1.9}$$

serves a the order parameter. Bottomline:

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[3]Do not palpitate. $T = \infty$ physics is same as any finite T physics with $J = 0$.

SYMMETRY AND PHASES

Symmetry can be used to distinguish phases. An ordered phase are described by a broken symmetry state, and a disordered phase by a symmetric state. Broken symmetry phase can by characterized by an order parameter.

Here I need to break the discussion of the Ising model for making an important point. Do not come away with the idea that phases are to be distinguished solely by symmetry. Nothing can be further from the truth. In fact, in many cases symmetry distinct phases will both have the same symmetry. Consider for example a simple two band tight binding model for spinless electrons. If the electron filling is such that the valance band if half filled, then we have *metallic* (liquid like) phase of the electrons, while if the filling is unity, then we have the band insulating (solid like) phase. In both cases the states will have all the symmetries of the system. What distinguishes these phases are their properties or response functions (i. e., difference between solid and liquid).

PHASES AND RESPONSE FUNCTIONS

Phases are not always distinguished by symmetry. Phases can be distinguished by their reponse functions.

Up until very recently (say about thirty years), it was thought that properties and symmetries are the sole criteria to distinguish phases. This has changed drastically in the last decade. We have realized that two additional concepts are crucially important. The first is notion of *entanglement* and the second is the notion of *topology*. We will not pause here to discuss these ideas, suffice it to mention

NEW DEVELOPMENTS

Notions of entanglement and topology are also essential to describe phases of systems with many degrees of freedom.

Gist of the discussion above (started with the Ising model), phases can be distinguished by symmetries, properties, entanglement and topolgy.

Back to the Ising model. We have seen that the Ising model has two phases. Can we change the phase from one to the other? The answer is of course, “yes”, as we all know, for example, by changing the temperature T. Suppose we assume that the phase change to occur at a critical temperature T_c, i. e., for $T < T_c$ the system is in the ordered phase while for $T > T_c$ it is in the disordered phase. There are a couple of possibilities. First possibility is that the phase change in discontinuous, called a *first order* transition. This is usually observed as a discontinuous jump in the order parameter at T_c, thus $M(T_c^-) \neq 0$ while $M(T_c^+) = 0$. The second possibility is of a *continuous phase transition*, where $M(T)$ is continuous at T_c, whth $M(T < T_c) \neq 0$ while $M(T \geq T_c) = 0$. In other words, $M(T)$ is continuous, but *not analytic* at T_c.

First order transitions occur when the free energy of the system has distinct local minima, the ordering of which changes at T_c – this is an analog of a quantum mechanical level crossing. Such transitions typically have a latent heat associated with them. Continuous transitions on the other hand arise form "truly cooperative" behaviour where the system organizes "scale by scale" as we approach T_c from either side. Can we tell a priori if our Ising model will have a first order or continuous transition? We know, of course, that the Ising model in spatial dimensions greater than 2 has a continuous transition at at $T_c > 0$. We know this from the exact solution of Onsager in 2 dimensions, and from numerical work (and experiment on Ising magnets) in 3 dimensions.

As you will be aware continuous transitions have rich physics in the some of which we will now review. We have already seen that the order parameter has a non-analytic behaviour near T_c. In fact, almost all interesting quantities have non-analytic character manifested as singularities in observables. To carry forward the discussion, let me introduce a dimensionless quantity t to take the discussion forward

$$t = \frac{T - T_c}{T_c}. \quad \text{eqn:tdef} \quad (1.10)$$

The specific heat suffers a divergence near T_c and behaves as

$$C_V = A_{\pm}^{C_V} |t|^{-\alpha} \quad (1.11)$$

where $A_{\pm}^{C_V}$ are quantities on the "right" and "left" of the transition. The order parameter, we have ready seen, behaves as

$$M = A_{\pm}^{M} |t|^{\beta} \quad (1.12)$$

where A_{+}^{M} is, obviously, zero. The magnetic susceptibility also diverges as

$$\chi = A_{\pm}^{\chi} |t|^{-\gamma} \quad (1.13)$$

At the critical point, the system has a nonlinear response to an external field. In the Ising case, we write

$$M = A^B B^{1/\delta} \quad (1.14)$$

Finally, away from T_c, correlations between spins on long scales $|x|$ behave as $e^{-|x|/\xi}/|x|^{d-2}$. The *correlation length* ξ characterizing the correlations between spins diverges as $T \to T_c$ as

$$\xi = A_{\pm}^{\xi}|t|^{-\nu}. \tag{1.15}$$

If you are seeing it for the first time, the most confounding aspect (shaking your very faith in sutff like dimensional analysis!) of continuous phase transitions is the correlation function between the spins at T_c

$$G(r) \sim \frac{1}{|x|^{d-2+\eta}}. \tag{1.16}$$

The quantities α, β, γ, δ, ν, η have been christened as *critical exponents*, and the quantities $A_{\pm}$ are called *amplitude factors*.

The most notable aspect of continuous transitions is that they have *universal physics* in them. To see what this means, let us first understand something that is *not* universal. Take, for example, T_c. T_c will depend strongly on the nature of the lattice – square lattice, triangular lattice, honeycomb lattice each will have its own T_c. On the same lattice, T_c will change if there are next neighbour interactions (recall that eqn. (1.1) had only nearest neighbour interactions.) Now for universal physics – what is found is that *critical exponents and amplitude ratios are* universal (note amplitude factors are *not* universal, the ratios A_+/A_- are), for the Ising model they depend only on the spatial dimension! What is even more striking is that it does not even matter *how exactly* the Ising symmetry is realized! The liquid-gas critical point (it takes some though to realize how the liquid-gas system realizes Ising symmetry) has *exactly the same* critical exponents as the magnetic critical point! In fact, this statement is true non-only for Ising systems, but systems with any symmetry. For a different symmetry, the critical exponents are generally different from the Ising case, but all systems that realize the given symmetry at a similar critical point will have the same long wavelength physics as characterized by the critical exponent. The key is to look at the long wavelength limit, i e., on (arena)scales much larger than the ultraviolet (lattice) scale. This usually goes under the name of a *scaling limit*. Lets box this:

UNIVERSAL PHYSICS IN THE SCALING LIMIT

Long wavelength physics near a critical point is universal in the sense that on scales much larger than the ultraviolet short distance lattice scale, physics is determined by a set of numbers (critical exponents and amplitude ratios) whose values are determined solely by things like symmetry and spatial dimension and not by the microscopic details.

Let me add a few important things to this discussion. First thing is that there are many universal critical exponents, perhaps this is a bit disconcerting already. Actually, the situation is not all that bad. The remarkable thing is that these exponents are not all independent. There are relationships between various exponents that are brought about by various things like conservation laws etc. For example,

$$\alpha + 2\beta + \gamma = 2 \quad \text{(Rushbrooke's Law)} \quad \text{eqn:Rushb} \tag{1.17}$$

$$\beta(\delta - 1) - \gamma = 0 \quad \text{(Widom's Law)} \quad \text{eqn:Widom} \tag{1.18}$$

$$\gamma - \nu(2 - \eta) = 0 \quad \text{(Fischer's Law)} \quad \text{eqn:Fish} \tag{1.19}$$

$$\nu d + \alpha = 2 \quad \text{(Hyperscaling law, not always valid)} \quad \text{eqn:Hyper} \tag{1.20}$$

In fact, at the critical point of the Ising model (which is called the Wilson-Fischer fixed point) there are only two independent critical exponents.

We will now go in a different direction, motivated by Landau and Ginzburg, and others. We will now write down a *field theory* in the usual sense of the term that captures the physics of the Ising model. Our arena is now a periodic chunk of a d-dimensional continuum space whose points are labelled by x. We know that our system has all the arena symmetries, and the $\mathbb{Z}_2$ symmetry. A theory is constructed by sticking at each point a real scalar field ϕ (the real numbers is a vector space on which the action group $\mathbb{Z}_2$ can be represented), the field being denoted by $\phi(x)$. We now write down a Hamiltonian density (to make contact with elementary statistical mechanics, what we write down must be really interpreted as the Hamiltonian density divide by temperature.)

$$H(x) = |\boldsymbol{\nabla}\phi|^2 + V(\phi^2) \quad \text{eqn:LGDens} \tag{1.21}$$

and

$$\mathcal{H}[\phi] = \int \mathrm{d}^d x \, H(x). \quad \text{eqn:LG} \tag{1.22}$$

The partition function is

$$Z = \int \mathcal{D}\phi \, e^{-\mathcal{H}[\phi]}. \tag{1.23}$$

Everything is written down to make the $\mathbb{Z}_2$ symmetry explicit.

Note that the theory in eqn. (1.22) is really a quantum field theory written in Euclidean time – and example of quantum classical correspondence.

Before discussing anything further, I want to point out a crucially important physical idea. Assuming that the volume of the d-dimensional periodic box is V, we can write

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_k \phi(k) e^{\mathrm{i} k \cdot x} \quad \text{eqn:finFT} \tag{1.24}$$

as a Fourier transform. At this stage the momentum k runs over all possible values, in fact, $k_\mu = \frac{2\pi n_\mu}{L_\mu}, \mu = 1, \ldots, d$ where n_μ is any integer and L_μ is the length of the μth edge of the box ($V = \prod_\mu L_\mu$). However, this cannot be physically meaningful! We really do not know the physics below some microscopic scale (atomic scale, if we are modeling Ising model physics with eqn. (1.22)). We must explicitly acknowledge this. This we do by introducing a *momentum cutoff* Λ into the theory (it is very reasonable to picture Λ to the of the order of the inverse lattice spacing of the Ising model discussed previously). Thus, in writing eqn. (1.22), what we really mean is that

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{|k| \leq \Lambda} e^{\mathrm{i} k \cdot x} \phi(k) \tag{1.25}$$

This is a generic idea, in that *every* field theory we write down will have an ultraviolet cutoff. One may interpret the cutoff as an explicit acknowledgement that we do not know physics at arbitrary small scales.

What can we say about the phases of the theory eqn. (1.22)? To answer this, we need some more information about $V(\phi^2)$. Let us write

$$V(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 \quad \text{eqn:Phi4} \tag{1.26}$$

taking $u > 0$ (for stability). We see immediately that $r = 0$ is special point (for a fixed u.) When $r > 0$, any nonzero value of $\phi(x)$ is energetically penalized by $V(\phi)$, but this is not the case when $r < 0$. When $r < 0$, the field ϕ itself may pick up an expectational value, and the cheapest field configuration that does this is constant field $\phi(x) = \phi_0 = \sqrt{-r/u}$. In this case, it is clear that the system breaks the $\mathbb{Z}_2$ symmetry.

Having established that the field theory eqn. (1.22) has an ordered phase $r < 0$ and a disordered phase $r > 0$, it is natural to ask about the point $r = 0$ which is the critical point. Based on our argument about universality, we realize that the physics of this critical point must be same as that of the Ising model in the scaling limit! This is already shows some very deep things, in particular the connection between statistical mechanics problems and quantum field theories.

Questions abound. Why do we have universality? "Where do the details of the system go?" How to obtain the universal physics (e.g. critical exponents) from theory? Wilson provided a solid framework that not only answers these questions, but also provides for a deeper understanding of what really a quantum field theory actually is. The framework is called the "renormalization group" (RG for short). We will visit some of these ideas next.

Wilson realized that everything that is "seen" about the critical point can be inferred if nature implements the following

"WILSON'S LAW"

Any system at a critical point possesses a much bigger symmetry. This additional symmetry is scale invariance*. Systems at a critical point are scale invariant.*

Many qualitative aspects discussed above follow from this "law". The natural question: what do we mean by scale invariance. Suppose, I am using a meter scale to measure units. Instead, I decide to use 10 meters, as the unit of measurement, and I *call* the old ten meters as a new one meter. Suppose, I look at the physics using this new length scale and "plot things out" in this new units, the polts can be made to look identical to what I had with the old units, provided I scale the quantity I am looking at by some factor. What is this factor? First let us call the scale factor s, in our example the scale factor $s = 10$. For *every* ("long wavelength") quantity Q, I will find that, if I scale the quantity Q by s^{d_Q}, then the plot in terms of the old units is identical to what I had in the old units. We say that d_Q is the scaling dimension of the quantity Q. A system is scale invariant, if every quantity "transforms by scaling" up on a scale transformation. In other words, under a scale transformation,

eqn:xscale

$$x_{\text{old}} \mapsto s\, x_{\text{new}} \tag{1.27}$$

all physical quantities are invariant ("plots in new units look same as old units after scaling appropriately")

$$Q_{\text{old}} \mapsto s^{d_Q}\, Q_{\text{new}} \tag{1.28}$$

In particular, under this transformation, eqn. (1.21) will transform as

$$H_{\text{new}}(x_{\text{new}}) = |\boldsymbol{\nabla}_{\text{new}}\phi_{\text{new}}|^2 + V_{\text{new}}((\phi^2)_{\text{new}}) = H_{\text{old}}(x_{\text{old}}) \tag{1.29}$$

i. e., the Hamiltonian is invariant under scale transformations! If we are close to, but not at the critical point, the Hamiltonian H_{new} will have the *same form* (even with some additional symmetry allowed terms), but with different parameters. To make this explicit: suppose we are not at the critical point and H is given by eqn. (1.21) and eqn. (1.26). Under the action of the RG transformation, H becomes ✍ **FIX NOTATION HERE**

$$H_{\text{new}} = |\boldsymbol{\nabla}_{\text{new}}\phi_{\text{new}}|^2 + P_1(s)|\boldsymbol{\nabla}_{\text{new}}\phi_{\text{new}}|^4 + P_2(s)\phi^2|\boldsymbol{\nabla}_{\text{new}}\phi_{\text{new}}|^2 + \ldots + V_{\text{new}}(\phi^2_{\text{new}}) \tag{1.30}$$

where $V_{\text{new}}(\phi^2_{\text{new}}) = \sum_{p=1}^{\infty} R_p(s, \{u_p\})(\phi^2_{\text{new}})^p$, where u_p stands for the original coefficients as in eqn. (1.26). In other words the RG transformation generates all terms allowed by symmetry.

Another way to view RG is as a map on a space that describes the Hamiltonian. Consider the space made of the coefficients and parameters that enters the Hamiltonian. RG maps an initial point P in this space to another denoted by

$$P \mapsto RG_s(P), \tag{1.31}$$

i. e. it produces a *flow*, usually called the RG flow. One constructs the RG map to have a semi-group structure

eqn:RGSG

$$RG_{s_2}(RG_{s_1}(P)) = RG_{s_1 s_2}(P) \tag{1.32}$$

Now consider the possibility of a *fixed point*, and call it $P^\star$ (there may be many fixed points; in fact such systems are the only interesting ones). Suppose P is a point in the vicinity of the fixed point $P^\star$ – will will denote $P = P^\star + \delta P$. Now consider a transformation with s close to unity. We expect

$$RG_s(P) \approx P^\star + L_s(P^\star)\delta P \tag{1.33}$$

where $L_s(P^\star)$ is a "linearized" version of RG transformation near $P^\star$. The next step is to find the "eigenvectors" of the linearized RG transformation. These eigenvectors denoted by O_i are "operators" that are "natural" at the fixed point $P^\star$ such that

$$L_s(P^\star)O_i = \lambda_i(s)O_i \tag{1.34}$$

From the semigroup structure of the renormalization group eqn. (1.32), it is immediate that

$$\lambda_i(s_1)\lambda_i(s_2) = \lambda_i(s_1 s_2) \tag{1.35}$$

and along with $\lambda_1(1) = 1$ suggests that

$$\lambda_i(s) = s^{\mu_i} \tag{1.36}$$

where μ_i is the scale dimension of the operator O_i. Now for an arbitrary δP, we write

$$\delta P = \sum_i g_i O_i \tag{1.37}$$

to get

$$L_s(P^\star) = \sum_i g_i s^{\mu_i} O_i \tag{1.38}$$

We see the key interesting point. If $\mu_i > 0$ the strength of the operator *increases* and the RG transformation takes us further away from the fixed point. Such operators are called *relevant* (superrenormalizable in field theory literature). If $\mu_i = 0$ this operator is called *marginal*. If $\mu_i < 0$, then the operator O_i is *irrelevant*. The final concept that we need is that of the critical manifold (or critical surface) near $P^\star$. The plane spanned by the marginal and irrelevant operators is *tangent to the critical surface*. Any starting Hamiltonian on the critical surface will stay on that surface under RG flow.

PHASES AND CRITICAL POINTS

A fixed point with no relevant operators describes a phase, while a fixed point with atleast one relevant operator is a critical point.

These ideas offer striking explanations of universality and scaling. The central point is that at any fixed point $P^\star$, nature allows *only* a handful of relevant operators. In other words, the critical surface at $P^\star$ is a very high dimensional manifold! This means most of the operators (interactions) in the Hamiltonian are irrelevant, and all of the physics at the critical points are determined by the relevant/marginal operators. This is why the details of the Hamiltonians wash away an physics is determined by a few parameters, the origin of universality.

Also, we see that the free energy density of a Hamiltonian near the critical point has the following property

$$f(\{g_i\}) = s^{-d} f(\{s^{\mu_i} g_i\}), \tag{1.39}$$

i. e., the free energy is a generalized homogeneous function of its parameters! The singular behaviour of thermodynamic quantities. The scaling laws also pop out! All critical exponents are determined solely by the scaling dimensions of *relevant* operators...so they are constrained, and this gives the scaling laws.

Wilson not only formulated these concepts, but also provided a calculational framework to implement the renormalization group transformation, something that has come to be known as Wilsonian RG in the literature. There are other methods: analytical ones include, so called, field theoretic RG, and then there is numerical RG (both of which Wilson himself contributed to). We will visit this to familiarize ourselves with some calculational techniques.

A key idea implicit in Wilson's approach (in contrast to what was in vogue in quantum filed theory) was an explicit acknowledgement and "respect" for the ultraviolet cutoff Λ. The essential question was how to implement a scale transformation that "does not change" the ultraviolet cutoff Λ. Here Wilson came up with an inspirational method, which not only bring in conceptual clarity but also formulates a nice calculational tool. Start with

$$Z = \int D[\phi] e^{-H[\phi]}, \qquad H[\phi] = \int \mathrm{d}^d x \, h[\phi] \tag{1.40}$$

eqn:GenHam

where $h[\phi]$ has, in principle, all terms in ϕ allowed by symmetries. The idea of defining RG_s, for $s > 1$ is to first write the fields as "slow" and "fast" variables

$$\phi(x) = \underbrace{\frac{1}{\sqrt{V}} \sum_{|k|<\frac{\Lambda}{s}} e^{\mathrm{i}kx} \phi(k)}_{\Phi(x)} + \underbrace{\frac{1}{\sqrt{V}} \sum_{\frac{\Lambda}{s}<|k|<\Lambda} e^{\mathrm{i}kx} \phi(k)}_{\psi(x)} \tag{1.41}$$

Now there is a three step process

Integrate: First is to "integrate out" the fast degrees of freedom to get a new action (or Hamiltonian) for the slow degrees of freedom. Technically

$$Z = \int D[\Phi] e^{-H_{\text{eff}}[\Phi]} = \int D[\Phi] \left[\int D[\psi] e^{-H[\Psi,\phi]} \right] \tag{1.42}$$

eqn:PartFunc

Note that at this stage the partition function is unchanged, and all that has happened in the "contribution" of fast degrees of freedom have been accounted for. Usually, this is the most technically challenging part.

Rescale: The previous step has effectively reduced the cutoff scale, Λ to $\frac{\Lambda}{s}$. We restore the cutoff scale by replacing $x \mapsto s x_{\text{new}}$ or $k \mapsto \frac{k_{\text{new}}}{s}$. With this rescaling, the cutoff momentum scale is restored back to Λ.

Renormalize: The final point (perhaps the most subtle) is to redefine the fields so as to "keep some term fixed" (more on this will be said later). This entails the following manipulations. First write (keep in mind, $x \mapsto sx_{\text{new}}$)

$$H_{\text{eff}}[\Phi] = \int \mathrm{d}^d x h_{\text{eff}}[\Phi] \tag{1.43}$$

Now define ϕ_{new} such that

eqn:RGphinew

$$\Phi(x) = \zeta(s)\phi_{\text{new}}(x_{\text{new}}) \tag{1.44}$$

where $\zeta(s)$ is the "field renormalization factor" (read on, don't stop here). We thus get

$$H_{\text{new}}[\phi_{\text{new}}] = \int \mathrm{d}^d x_{\text{new}} h_{\text{new}}[\phi_{\text{new}}] \tag{1.45}$$

where,

eqn:RGhtrans

$$h_{\text{new}}[\phi_{\text{new}}] = s^d h_{\text{eff}}[\zeta(s)\phi_{\text{new}}] \tag{1.46}$$

The key point about renormalization is that $\zeta(s)$ is *chosen* so as to "keep some term the same" in h_{new} and h. Stated in other words, if we start with h keeping all terms in ϕ allowed by symmetry as in eqn. (1.40), then h_{new} is another such expression with new coefficients which depend, in principle, on the old coefficients and, in addition, on s. The idea of renormalization is to choose the factor $\zeta(s)$ such that the coefficient of one term (for example, the coefficient of $(\boldsymbol{\nabla}\phi)^2$, as we shall do later) is left invariant under the transformation. The map $h \mapsto h_{\text{new}}$ defines RG_s.

A fixed point now corresponds to the condition of equality of two functions $h_{\text{new}}[\cdot] = h[\cdot]$.

Lets get some action. Consider the following theory

$$h[\phi] = (\boldsymbol{\nabla}\phi)^2 + t\phi^2 - B\phi \tag{1.47}$$

in eqn. (1.40), where I have also added as symmetry breaking field B. The free energy density of this theory can be calculated exactly. We obtain the free energy density as

$$f(t, B) = -\frac{B^2}{4t} + \frac{1}{(2\pi)^d} \int_{|k|<\Lambda} \mathrm{d}^d k \ln\left(k^2 + t\right) \tag{1.48}$$

The specific heat is two derivatives of the free energy at zero magnetic field. We get

$$c_v = -\frac{\partial^2 f(t,0)}{\partial t^2} = \frac{1}{(2\pi)^d}\int_{|k|<\Lambda} \mathrm{d}^d k \frac{1}{(k^2+t)^2} = K_d \int_0^\Lambda \mathrm{d}k \, \frac{k^{d-1}}{(k^2+t)^2} \sim t^{\frac{d-4}{2}} \tag{1.49}$$

a result which is cutoff independent for $d < 4$. Here and henceforth

$$K_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \tag{1.50}$$

eqn:Kddef

All this indicates that

$$\alpha = \frac{4-d}{2}. \tag{1.51}$$

Next we see that the susceptibility

$$\chi = \frac{1}{2t} \Longrightarrow \gamma = 1. \tag{1.52}$$

Lets look at the correlation function

$$\langle \phi(x)\phi(0)\rangle = \frac{1}{(2\pi)^d}\int_{|k|<\Lambda} \frac{e^{\mathrm{i}kx}}{k^2+t} \sim \frac{e^{-|x|/t^{-1/2}}}{|x|^{d-2}} \tag{1.53}$$

We see two things. The correlation length ξ is

$$\xi = t^{-1/2} \Longrightarrow \nu = \frac{1}{2} \tag{1.54}$$

and at $t = 0$,

$$\langle \phi(x)\phi(0)\rangle \sim \frac{1}{|x|^{d-2}} \Longrightarrow \eta = 0. \tag{1.55}$$

We see, reassuringly, that Fischer eqn. (1.19) is satisfied. I may add here that exponents δ and β are not quite meaningful for this theory.

The above exact solution seems to suggest that $t = 0$ point is a critical point of the theory as many interesting quantities are diverging there. Can we see this from the RG? Lets turn the crank. First identify $s > 1$ and split into fast and slow fields

$$\phi(x) = \Phi(x) + \psi(x) \tag{1.56}$$

The energy density becomes

$$h[\Phi,\psi] = (\boldsymbol{\nabla}\Phi)^2 + 2\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\psi + (\boldsymbol{\nabla}\psi)^2 + t(\Phi^2 + 2\Phi\psi + \psi^2) \tag{1.57}$$

It is easy (should be) to see that

$$h_{\text{eff}}[\Phi] = (\boldsymbol{\nabla}\Phi)^2 + t\Phi^2. \tag{1.58}$$

We now see that

$$h_{\text{new}}[\phi_{\text{new}}] = s^d s^{-2}\zeta^2(s)(\boldsymbol{\nabla}_{\text{new}}\phi_{\text{new}})^2 + s^d\zeta^2(s)t\phi_{\text{new}}^2 \tag{1.59}$$

Now the renormalization condition is that the coefficient of $(\boldsymbol{\nabla}\phi)^2$ term does not change. This gives

eqn:FFscale

$$\zeta(s) = s^{\frac{2-d}{2}} \tag{1.60}$$

resulting in

$$h_{\text{new}}[\phi_{\text{new}}] = (\boldsymbol{\nabla}_{\text{new}}\phi_{\text{new}})^2 + s^2t, \phi_{\text{new}}^2 \tag{1.61}$$

In other words, RG_s acting on the Gaussian model again gives a Gaussian model with new couplings $t \mapsto s^2t$. For the fixed point we demand

$$RG_s(t) = t, \Longrightarrow s^2t = t \tag{1.62}$$

This is satisfied by two values of t, $t = 0$ and $t = \infty$. Lets focus on $t = 0$ fixed point. Is $t > 0$, then under RG, t will grow, i. e., t is a relevant operator. So $t = 0$ fixed point is a critical point, and must have a diverging lengths scale. It is easy to see that

$$\xi \sim t^{-1/2} \tag{1.63}$$

and $\nu = 1/2$ as obtained in the analytical solution. Also $\eta = 0$ is immediate. From the scaling law eqn. (1.20) $\gamma = 1$, and from eqn. (1.20), we reproduce the analytical result of α. In other words, RG was able to provide us everything!

Ambition swells. What can RG tell us about an interacting theory, we are impatient. But wait, there is more to understand in the Gaussian model. We can ask some simple questions. Why did t map to s^2t, why is $\zeta(s) = s^{\frac{2-d}{2}}$? Note that H in eqn. (1.40) is dimensionless. In other words, since $H \sim L^0$ (L is length), its dimensions are

$$[\![H]\!] = 0 \implies [\![h]\!] = -d \tag{1.64}$$

leading to

$$[\![\phi(x)]\!] = \frac{2-d}{2} \tag{1.65}$$

and

$$[\![t]\!] = 2! \tag{1.66}$$

These are the *engineering dimensions* of the objects involved. We see that the critical exponents of the gaussian model are determined by the engineering dimension of the relevant operator! This is some what disconcerting...all this work to find the engineering dimensions, something that we learnt in 11th standard!

1.0.1 Large N

We will now show how to approach the physics by considering models with $O(N)$ symmetry, particularly focusing on cases where N is large. The canonical model for discussing this has a $O(N)$ vector $\boldsymbol{\phi} \equiv (\phi_1, \phi_2, \ldots, \phi_N)$, with a Hamiltonian density

$$h[\boldsymbol{\phi}] = (\boldsymbol{\nabla}\boldsymbol{\phi})^2 + t\boldsymbol{\phi}^2 + \frac{u}{2}(\boldsymbol{\phi}^2)^2 - B\phi_1 \tag{1.67}$$

The partition function

$$Z = \int D[\boldsymbol{\phi}]\, e^{-\int \mathrm{d}^d x h[\boldsymbol{\phi}(x)]} \tag{1.68}$$

First lets try a cheap stunt. Suppose, we say that the physics physics is solely determined by the uniform expectation value of $\langle \boldsymbol{\phi} \rangle = \boldsymbol{\phi}_\mathrm{L}$ (L stands for Landau), we see that the energy density is minimized if

$$(t + u|\boldsymbol{\phi}_\mathrm{L}|^2)\phi_{\mathrm{L}a} = B\delta_{a1} \tag{1.69}$$

Resulting in

$$|\boldsymbol{\phi}_\mathrm{L}| = \Theta(-t)\sqrt{\frac{|t|}{u}}, \qquad (B = 0) \tag{1.70}$$

leading to the Landau value of the exponent

$$\beta_\mathrm{L} = \frac{1}{2}. \tag{1.71}$$

Also, at $t = 0$,

$$\phi_{\mathrm{L}1} = \left(\frac{B}{u}\right)^{1/3} \tag{1.72}$$

given

$$\delta_\mathrm{L} = 3 \tag{1.73}$$

One also sees that

$$\phi_{\mathrm{L}a} = \frac{B}{t}\delta_{a1}, \quad t > 0 \tag{1.74}$$

leading to

$$\gamma_{\mathrm{L}} = 1. \tag{1.75}$$

By analysing the free energy, one also find that the specific heat has a jump at $t = 0$ leading to

$$\alpha_{\mathrm{L}} = 0. \tag{1.76}$$

Too cheap...critical exponents do not depend on N (on the symmetry), or the spatial dimension d.

One can try a bit harder by including Gaussian fluctuations about ϕ_L. As an exercise, you can show that nothing really changes, except α, which becomes

$$\alpha_{\text{gaussian}} = \frac{4-d}{2}. \tag{1.77}$$

(Question: Are scaling laws okay?)

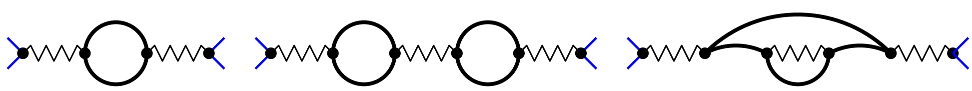
Not satisfactory!

We will now try a different route, and try and solve the problem "exactly", but for this we have to pose the question in a suitable fashion. Dipping into the work of stalwarts, we realize that an exact solution if feasible when $N \to \infty$, i. e., when the vector field ϕ has a very large number of components, aka "flavors". The key starting point of this discussion is the observation that in the large-N limit the field $\phi^2(x)$ (recall $\phi^2 = \phi \cdot \phi$), becomes "classical" if the problem is defined appropriately; i. e.,

eqn:Phi2

$$\langle \phi^2(x) \cdot \phi^2(x) \rangle \quad \approx_{N\to\infty} \quad \langle \phi^2(x) \rangle \langle \phi^2(y) \rangle. \tag{1.78}$$

What on earth does "appropriately" mean? To see see write one loop perturbative expansion of the lhs of eqn. (1.78), and look at a few of "quantum corrections":



One sees a generic pattern – for a correction at the m-th order with ℓ loops, the term goes as $u^m N^\ell$ as the solid bold lines are integrated over all flavours. For example, the diagrams shown above contribute, respectively, $u^2 N$, $u^3 N^2$ and $u^3 N$. In fact, it can be easily seen that at mth order, the diagram with the largest number of loops is when $\ell = m - 1$; the first two are the diagrams of this series that contribute, i. e., the "largest" mth order quantum correction goes as $u^m N^{m-1}$. We see an opportunity...suppose we redefine

$$u \mapsto \frac{u}{N} \tag{1.79}$$

we see that the leading contribution at the m-th order goes as $\frac{u^m}{N}$. As is evident, this vanishes in the limit of $N \to \infty$, as do all other contributions. A little thought will reveal that we can work with a much more general scenario where $t\boldsymbol{\phi}^2 + \frac{u}{2N}(\boldsymbol{\phi}^2)^2$ is replaced by a more general "potential" function

$$U(x) = \sum_{p=1}^{\infty} u_p x^p \tag{1.80}$$

with u_ps as the coupling constants, where we use $\boldsymbol{\phi}^2/N$ as the argument. Only condition we need is that $U(x) \to \infty$ as $x \to \infty$.

We will now put down the theory with appropriate changes

eqn:ONmodel

$$h[\boldsymbol{\phi}] = (\boldsymbol{\nabla}\boldsymbol{\phi})^2 + NU\left(\frac{\boldsymbol{\phi}^2}{N}\right) - \sqrt{N}B\phi_1 \tag{1.81}$$

where we have scaled B by an appropriate power of N anticipating the future. Note, symmetry has not been touched, only redefinition of coupling constants. Observation eqn. (1.78) now suggests that $\boldsymbol{\phi}^2(x)$, put colloquially, takes a life of its own and can be thought of as another field $\sigma(x)$, i. e., we want to impose

$$\boldsymbol{\phi}^2(x) \text{ `` = '' } N\sigma(x) \tag{1.82}$$

where the factor of N anticipates the future. We do this in the following way

eqn:LNstart

$$Z = \int D[\boldsymbol{\phi}]e^{-\int \mathrm{d}^d x\, h[\boldsymbol{\phi}]} = \int D[\boldsymbol{\phi}]D[\sigma]\,\Delta[\boldsymbol{\phi}^2(x) - N\sigma(x)]e^{-\int \mathrm{d}^d x\, h[\boldsymbol{\phi}]} \tag{1.83}$$

where Δ is the functional Dirac delta function. We now write the functional Dirac delta by introducing another field $\lambda(x)$ as

$$\Delta[\boldsymbol{\phi}^2(x) - N\sigma(x)] = \int D[\lambda]e^{-\mathrm{i}\lambda(x)(\boldsymbol{\phi}^2(x) - N\sigma(x))} \tag{1.84}$$

after which we get

$$Z = \int D[\boldsymbol{\phi}]D[\sigma]D[\lambda]\, e^{-\int \mathrm{d}^d x h[\boldsymbol{\phi},\sigma,\lambda]} \tag{1.85}$$

where, after some rearrangement,

$$h[\boldsymbol{\phi},\sigma,\lambda] = \boldsymbol{\phi}\cdot(-\boldsymbol{\nabla}^2 + \mathrm{i}\lambda)\boldsymbol{\phi} + NU(\sigma) - \mathrm{i}\lambda\sigma - \sqrt{N}B\phi_1 \tag{1.86}$$

Analysis proceeds as follows. First redefine

$$\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_N) = (\varphi, \psi_1, \ldots, \psi_{N-1}) = (\varphi, \boldsymbol{\psi}). \tag{1.87}$$

It is easy to see that $\boldsymbol{\psi}$ field talks only to the λ field, and can be integrated out leading to a term, $(N-1)\ln\det[-\boldsymbol{\nabla}^2 + \mathbb{i}\lambda]$, resulting in

$$h[\varphi,\sigma,\lambda] = \varphi(-\boldsymbol{\nabla}^2 + \mathbb{i}\lambda)\varphi + NU(\sigma) - \mathbb{i}\lambda\sigma - \sqrt{N}B\varphi + N\ln\det[-\boldsymbol{\nabla}^2 + \mathbb{i}\lambda] \tag{1.88}$$

As physicists, we do not notice the difference between $N-1$ and N! Now, a matter of redefinition is in order. Define,

$$\varphi = \sqrt{N}\Phi \tag{1.89}$$

resulting in

$$Z = \int D[\Phi]D[\sigma]D[\lambda]\, e^{-NH[\Phi,\sigma,\lambda]}, \quad H[\Phi,\sigma,\lambda] = \int \mathrm{d}^d x\, h[\Phi,\sigma,\lambda] \tag{1.90}$$

eqn:LNfin

where

$$h[\Phi,\sigma,\lambda] = \Phi(-\boldsymbol{\nabla}^2 + \mathbb{i}\lambda)\Phi + U(\sigma) - \mathbb{i}\lambda\sigma - B\Phi + \ln\det[-\boldsymbol{\nabla}^2 + \mathbb{i}\lambda]. \tag{1.91}$$

eqn:LNh

Lets consolidate – we did two things in going from eqn. (1.83) to eqn. (1.90). First, the important one, we made $\boldsymbol{\phi}^2$ a new field $N\sigma(x)$, and second we replaced $N-1$ by N!

Eqn. (1.90) now allows us a very nice starting point for further analysis. Since $N \to \infty$, the path integral will be dominated by its "classical" value, i. e., where the fields extremize the action H in eqn. (1.90). We look for classical field that are "constant", i. e., independent of the arena point x. This leads to the following "uniform saddle point" equations

eqn:PhiSP

$$2\mathbb{i}\lambda\Phi = B \tag{1.92}$$

eqn:sigSP

$$-\mathbb{i}\lambda + U'(\sigma) = 0 \tag{1.93}$$

eqn:lamSP

$$\Phi^2 - \sigma + \int \mathrm{d}^d k \frac{1}{k^2 + \mathbb{i}\lambda} = 0 \tag{1.94}$$

Suppose we now let $B = 0$. Then the eqn. (1.92) says that either $\Phi = 0$ or $\mathbb{i}\lambda = 0$. To understand what this means, it is best to redefine the saddle point $\mathbb{i}\lambda$ as m^2, i. e., "mass". One finds that if $\Phi \neq 0$, which means that we break $O(N)$ symmetry, $m^2 = 0$ necessarily. We see that in the situation where $\Phi \neq 0$, the $\boldsymbol{\psi}$ fields are gapless! This is a manifestation of the *Goldstone theorem*; in a system with continuous symmetry and local interactions, breaking of the symmetry will result in a state with gapless excitations!

To proceed further, we need to study the integral.

$$\int \mathrm{d}^d k \frac{1}{k^2+m^2} \tag{1.95}$$

This integral is ultraviolet divergent for $d-3>-1$, i. e, for $d>2$. Also, there are infrared problems if $d-3<-1$, i.e., $d<2$. We now introduce a cutoff momentum Λ. We see that

$$\begin{aligned} F(\Lambda, m^2) = \int_{|k|<\Lambda} \mathrm{d}^d k \frac{1}{k^2+m^2} &= \frac{K_d}{d-2}\Lambda^{d-2} - m^2 K_d \int_0^\Lambda \mathrm{d}k \frac{k^{d-3}}{k^2+m^2} \\ &\approx \frac{K_d}{d-2}\Lambda^{d-2} - m^{d-2}K_d f(d) + \frac{m^2 K_d}{d-4}\Lambda^{d-4} \end{aligned} \tag{1.96}$$

eqn:FLambdam

where (the second integral is convergent for $d<4$, and for $d=4$ has a log divergence)

$$f(d) = \frac{\pi}{2\sin\frac{\pi(d-2)}{2}}. \tag{1.97}$$

Note that the last term vanishes for $d<\infty$ for "large" λ. Main point, in our regime of interest $2<d<4$, only the first two terms of $F(\Lambda, m^2)$ are important.

At the critical point, both $\Phi=0$ and $m^2=0$. We get, then that

$$\sigma_c = \frac{1}{d-2}\Lambda^{d-2} \tag{1.98}$$

From eqn. (1.93), we get, for special $U(x)$, that

$$t_c + u\sigma_c = 0 \tag{1.99}$$

Since in the broken symmetry phase ($m^2=0$),

$$t + u\sigma = 0 \tag{1.100}$$

whence

$$t_c - t = u(\sigma - \sigma_c) \tag{1.101}$$

Using eqn. (1.94), we get

$$\Phi = \sqrt{\frac{t_c - t}{u}} \tag{1.102}$$

or

$$\beta = \frac{1}{2} \tag{1.103}$$

In the disordered phase ($\Phi = 0$), we get from eqn. (1.94) using eqn. (1.96)

$$\sigma_c - \sigma = m^{d-2} K_d f(d) \tag{1.104}$$

or,

$$m^2 = \xi^{-2} = \left(\frac{t - t_c}{u K_d f(d)} \right)^{\frac{2}{d-2}} \tag{1.105}$$

We thus get

$$\nu = \frac{1}{d-2} \tag{1.106}$$

which (yippie!) is *different* from the Landau value, and is dimension dependent. In the disordered phase, the correlator look like

$$\langle \phi_a(x) \phi_b(0) \rangle = \delta_{ab} \frac{e^{-|x|/\xi}}{|x|^{d-2}} \tag{1.107}$$

which immediately gives

$$\eta = 0 \tag{1.108}$$

Finally, we see that in the normal phase

$$\chi = \frac{1}{2m^2} \sim (t - t_c)^{-\frac{2}{d-2}} \tag{1.109}$$

or

$$\gamma = \frac{2}{d-2} \tag{1.110}$$

One can easily see that one gets obtains Fisher scaling law eqn. (1.19) okay! Next, using eqn. (1.17), we find that

$$\alpha = \frac{d-4}{d-2} \tag{1.111}$$

which is also consistent with hyperscaling law eqn. (1.20).

From eqn. (1.91), we can obtain an expression for the saddle point density as

$$f = m^2(-F(\Lambda, m)) + U(\sigma) - B\Phi + \frac{K_d}{2} \Lambda^{d-2} m^2 - \frac{K_d f(d)}{d} m^d \sim (t - t_c)^{\frac{d}{d-2}} \tag{1.112}$$

which again gives the value of α, just quoted.

Finally, at $t = t_c$, we get $\Phi^2 = K_d f(d) m^{d-2}, 2m^2\Phi = B$, resulting in

$$\Phi^{\frac{d+2}{d-2}} \sim B \tag{1.113}$$

implying the critical exponent

$$\delta = \frac{d+2}{d-2} \tag{1.114}$$

One is relieved that Widom eqn. (1.18) has not been left behind.

1.0.2 Techniques: Dimensional Regularization

If you look at certain literature, you will be faced with phrases like "carrying out the RG procedure in the dimensional regularization scheme with minimal subtraction", and if you read on you will will be faced with extremely strange results such as

$$\int_0^\infty \mathrm{d}k k^{d-1} = 0 \tag{1.115}$$

and the only reason why you were cowed down was because this is the Veltman formula. You know better than to disparage a Nobel Prize winner.

Time to pull up our socks and get on with it. We need some revision. Recall the definition of the gamma function

eqn:Gamma

$$\Gamma(z) = \int_0^\infty \mathrm{d}x\, e^{-x} x^{z-1} \tag{1.116}$$

An important result is

$$\Gamma(1) = 1. \tag{1.117}$$

It is also easy to see that

$$z\Gamma(z) = \Gamma(z+1) \tag{1.118}$$

providing the result

$$\Gamma(n) = (n-1)! \tag{1.119}$$

One sees that the gamma function is the generalization of the notion of factorials to an arbitrary complex number. The gamma function is positive for positive arguments of z, and has singularities at all non positive integers. In fact for any $n \geq 0$, we have

eqn:Expn

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + O(\epsilon)\right] \tag{1.120}$$

where

eqn:DiGamma

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.121}$$

is the digamma function, which, again, is positive for positive argument.

Another key function to recall is the beta function

$$\mathcal{B}(x,y) = \int_0^1 \mathrm{d}t\, t^{x-1}(1-t)^{y-1} \tag{1.122}$$

A neat result is that

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.123}$$

eqn:Beta

Colloquially, the beta function is the generalization of the combinatorial ${}^nC_r$ function.

Now we will state the all important formula for us

$$\int_0^\infty \mathrm{d}x\, x^a(1+x^2)^{-b} = \frac{1}{2}\mathcal{B}\left(\frac{a+1}{2}, b - \left(\frac{a+1}{2}\right)\right) \tag{1.124}$$

eqn:Imp

It is not very difficult to prove this from the definitions.

In our calculations, we need

$$\begin{aligned}\int \mathrm{d}^d k \frac{1}{(k^2+m^2)^b} &= \frac{2m^{d-2b}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \mathrm{d}k\, k^{d-1}(1+k^2)^{-b} \\ &= \frac{m^{d-2b}}{(4\pi)^{d/2}} \frac{\Gamma(b-\frac{d}{2})}{\Gamma(b)}\end{aligned} \tag{1.125}$$

eqn:Kint

✍ **(Here $\mathrm{d}^d k$ *includes* the factor of $1/(2\pi)^d$)** Note there are no explicit cutoffs anywhere. The central idea is that the dimension d has been made an arbitrary "complex" number. The divergent integrals that we faced earlier will appear as poles of the gamma function. This trick leads to what is called as "dimensional regularization". The remarkable thing about this is that there is no explicit cutoff, and stated in a very field theoretic fashion, the theory remains Lorentz invariant.

There are some beautiful electrostatic examples for the use of dimensional regularization **[?, ?]**.

Let us work with the $O(N)$ ϕ^4 model,

$$h[\boldsymbol{\phi}] = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\phi})^2 + \frac{1}{2}m^2\boldsymbol{\phi}^2 + \frac{u}{2N}(\boldsymbol{\phi}^2)^2 \tag{1.126}$$

eqn:ONmodelDR

to learn how dimensional regularization works. Before proceeding, we introduce a mass scale μ, and write

$$u = g\mu^{4-d} \tag{1.127}$$

where g is a *dimensionless* coupling constant.

At the one loop level the self energy diagrams are

The first one has a "topological factor" (tf) of 4, while the latter 8, leading to

$$\Sigma^{(1)}(k) = -2u\frac{(N+2)}{N}\int \mathrm{d}^d k \frac{1}{k^2+m^2} = -\frac{2u(N+2)}{N}\frac{m^{d-2}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right) \tag{1.128}$$

a term that is k independent. The calculation must agree with observation

$$m^2_{\text{observed}} = m^2 + \frac{2g\mu^{4-d}(N+2)}{N}\frac{m^{d-2}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right) \tag{1.129}$$

eqn:BlowUp

but the rhs is divergent in $d=4$! Our theory seems to make no sense, is it nonsense? Let us work near 4 dimensions, $d = 4-\epsilon$. We get that

$$m^2_{\text{obs}} = m^2\left(1+\frac{2g(N+2)}{N(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}}\Gamma\left(-1+\frac{\epsilon}{2}\right)\right) \tag{1.130}$$

With dimensional regularization, we have "controlled" the infinities (i. e., we know how to parameterize the infinity). Yet, this leaves a very simple question – what the frog is going on?

✍ **Write out an elementary introduction to perturbation theory.** Turns out that we have to go back to the drawing board, and revisit quantum field theory again! Of course, we have set up everything, but not one little thing before starting off. Since our fields are $O(N)$ object, they carry a vector index $\phi_a(x)$. We will, as far as as possible, never explicitly write these indices anywhere. An interesting thing to calculate is the n-point correlator

$$G^{(n)}(\{x_i\}) = \langle\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = \frac{\int D\boldsymbol{\phi}\,\phi(x_1)\phi(x_2)\dots\phi(x_n)\,e^{-\int \mathrm{d}^d x h[\boldsymbol{\phi}]}}{\int D\boldsymbol{\phi}\,e^{-\int \mathrm{d}^d x h[\boldsymbol{\phi}]}} \tag{1.131}$$

eqn:Gn

The denominator is simply the partition function. We can now expand the numerator as a perturbation series in u, and that the all terms with vacuum diagrams multipliying an otherwise connected diagram are cancelled by the denominator. Lets define things in Fourier space (in a slightly different fashion than eqn. (**??**)) as

$$f(k) = \int \mathrm{d}^d x\, e^{-\mathrm{i}kx} f(x) \tag{1.132}$$

and the transform as

$$f(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, e^{\mathbb{i}kx} f(k) \tag{1.133}$$

We have implicit, the formula

$$\int \mathrm{d}^d x \, e^{-\mathbb{i}kx} = (2\pi)^d \delta(k). \tag{1.134}$$

The correlator $G^{(n)}$ in Fourier space looks like

$$G^{(n)}(k_1, \ldots, k_n) = \int \prod_{i=1}^{n} \mathrm{d}^d x_i \, e^{-\mathbb{i}\sum_i k_i x_i} G^{(n)}(x_1, \ldots, x_n) \tag{1.135}$$

In a translationally invariant state, we expect the following structure

$$G^{(n)}(k_1, \ldots, k_n) = (2\pi)^d \delta \left(\sum_i k_i \right) \bar{G}^{(n)}(k_1, \ldots, k_n) \tag{1.136}$$

Now, focussing, on the connected pieces of $\bar{G}^n_c$, it is useful to define the vertex

$$\bar{\Gamma}^{(n)}(k_1, \ldots, k_2) = \frac{\bar{G}^{(n)}_c(k_1, \ldots, k_n)}{\bar{G}^{(2)}(k_1, -k_1) \ldots \bar{G}^{(2)}(k_n, -k_n)} \tag{1.137}$$

In other words, $\bar{\Gamma}^{(n)}$ is an irreducible vertex with n external lines, and is obtained from the connected correlator $\bar{G}^{(n)}_c$ by "clipping" the n external $\bar{G}^{(2)}$ lines.

We will need one more set of correlation function, the ones that involve the so called "composite operators". We write

$$G^{(l,n)}(y_1, \ldots, y_l, x_1, \ldots, x_n) = \frac{1}{2^l} \langle \phi^2(y_1) \ldots \phi^2(y_l) \phi(x_1) \ldots \phi(x_n) \rangle \tag{1.138}$$

The Fourier transform of this is naturally defined as

$$G^{(l,n)}(q_1, \ldots, q_l, k_1, \ldots, k_n) \tag{1.139}$$

A key result is the following

$$G^{(1,n)}(q = 0, k_1, \ldots, k_n) = FT_{x_i} \left[\frac{1}{2} \int \mathrm{d}^d y G^{(1,n)}(y, x_1, \ldots, x_n) \right] \tag{1.140}$$

where FT is abbreviation of Fourier transform. A useful result is that

$$G^{(1,n)}(q = 0, x_1, \ldots, x_n) = -\frac{\partial}{\partial m^2} G^{(n)}(x_1, \ldots, x_n) \tag{1.141}$$

and

$$G^{(1,n)}(q=0,k_1,\ldots,k_n) = -\frac{\partial}{\partial m^2} G^{(n)}(k_1,\ldots,k_n) \tag{1.142}$$

Before we proceed with the analysis, let us recall a few basic facts. For the theory defined in eqn. (1.126), we have

$$[\![m]\!] = L^{-1} = \mu^1, [\![\phi(x)]\!] = L^{-D_\phi} = \mu^{D_\phi}, [\![u]\!] = L^{d-4} = \mu^{4-d} \tag{1.143}$$

where μ is the previously introduced mass scale. The engineering dimension of the field $\phi(x)$ is

$$D_\phi = \frac{d}{2} - 1 \tag{1.144}$$

With this discussion it is immediately clear that

$$\begin{aligned} [\![G^{(l,n)}(x)]\!] &= L^{-(2l+n)D_\phi} = \mu^{(2l+n)D_\phi} \\ [\![G^{(l,n)}(k)]\!] &= L^{-(2l+n)D_\phi+(l+n)d} = \mu^{l(2D_\phi-d)+n(D_\phi-d)} \end{aligned} \tag{1.145}$$

Finally, we note that

$$[\![\bar{\Gamma}^{(n)}(k)]\!] = \mu^{-(nD_\phi-d)} \tag{1.146}$$

eqn:GamDim

One can now write formal perturbation expansion for $\bar{\Gamma}^{(n)}$ in powers of g. As we have seen, this will produce infinite results!

Our idea now is extract as much as possible without explicit calculation. ✍ **Citation** We first ask, what is the consequence of Wilson's law (see above eqn. (1.27)) on the correlation functions. Wilson's law suggests that the system has an emergent symmetry. In such cases, the correlation functions are not all independent – there will be constraints imposed by the symmetry. Such constraints are generically termed as Ward identities. An example of this is the Einstein relation connecting the conductivity with the diffusion constant, a result of particle number conservation. What conditions does scale invariance or dialatational symmetry impose on our system?

To investigate this, we first set $m = 0$ and $g = 0$ obtaining a free field theory. Suppose we perform the scale transformation eqn. (1.27), to get (recall analog of eqn. (1.44))

$$G^{(n)}(x) = \zeta(s)^n G^{(n)}_{\text{new}}(x_{\text{new}}) \tag{1.147}$$

For the free field, we have seen that $\zeta(s)$ satisfies eqn. (1.60)

$$G^{(n)}(x) = s^{-nD_\phi`} G^{(n)}_{\text{new}}(x_{\text{new}}) \tag{1.148}$$

a result that is true for all s. To make progress, define $s = e^{\ell}$, and ask what happens for infinetisimal ℓ. The ideas that $G^{(n)}_{\text{new}}$ as a function of its arguments is same as the function $G^{(n)}$ implies that

$$G^{(n)}(\{x\}) = (1 - nD_\phi \ell)\left(G^n(\{x\})) - \ell \sum_i x_i \partial_{x_i} G^n(\{x\})\right) \tag{1.149}$$

Scale invariance now gives the Ward identity

$$\sum_i x_i \partial_{x_i} G^{(n)}(x) + nD_\phi G^n(\{x\}) = 0 \tag{1.150}$$

In other words, all correlation functions are eigenfunctions of the dilatation operator with an eigenvalue determined by the engineering dimensions. Applying this to $G^{(2)}(x_1, x_2)$ along with translational symmetry immediately gives G to be a homogeneous function of degree $-nD_\phi$ and in fact a power law in the argument

$$G^{(2)}(x_1 - x_2) \sim |x_1 - x_2|^{-(D-2)} \tag{1.151}$$

a very well known result. One can immediately obtain from this a k-space version of the Ward identity as

$$\sum_i k_i \partial_{k_i} G^n(\{k_i\}) = (nD_\phi - nd) G^n(\{k_i\}) \tag{1.152}$$

(This is obtained by noting that $FT(x\partial_x f) = i\partial_k(FT(\partial_x f)) = -\partial_k(kf(k)) = -k\partial_k f(k) - df(k)$) We can go on to obtain

$$\left(\sum_i k_i \partial_{k_i}\right) \bar{\Gamma}^{(n)}(\{k_i\}) = (-nD_\phi + d)\bar{\Gamma}^{(n)}(\{k_i\}) \tag{1.153}$$

a result that is consistent with dimensional consideration eqn. (1.146). This is nice since $\bar{\Gamma}^2(k, -k) \sim k^2$ for the free field.

Okay, what happens if we add a mass term to the theory? Scale invariance is clearly broken, but can we be more specific about this? Here is a nice trick to answer this question. We want to calculate $G^{(n)}_{new} - G^{(n)}$ to linear order in ℓ. Suppose we map from $x \to x_{\text{new}}$ via eqn. (1.27), where $s = e^{\ell}$. Under this map, an operator O changes from $O + \ell\delta O$, the action changes from $H + \ell\delta H$. Thus, we get that

eqn:deltaO

$$\delta\langle O\rangle = \ell\left[\langle\delta O\rangle - (\langle\delta H\, O\rangle - \langle\delta H\rangle\langle O\rangle)\right] \tag{1.154}$$

Suppose O is

$$O = \phi(x_1)\phi(x_2)\ldots\phi(x_n), \tag{1.155}$$

then

$$\delta O = -(nD_\phi + \sum_i x_i \partial_{x_i})O \tag{1.156}$$

Also,

$$\delta H = -2m^2 \underbrace{\int \mathrm{d}^d x \frac{1}{2} m^2 \boldsymbol{\phi}^2(x)}_{\delta H_{m^2}} - (4-d)u \underbrace{\int \mathrm{d}^d x \frac{1}{2} (\boldsymbol{\phi}^2(x))^2}_{\delta H_u} \tag{1.157}$$

Now, one can show that

$$\begin{aligned} \langle \delta H_{m^2} O\rangle - \langle \delta H_{m^2}\rangle\langle O\rangle &= \frac{\partial}{\partial m^2}\langle O\rangle \\ \langle \delta H_u O\rangle - \langle \delta H_u\rangle\langle O\rangle &= \frac{\partial}{\partial u}\langle O\rangle \end{aligned} \tag{1.158}$$

With these developments, we find that for scale invariance,

$$\left[-(nD_\phi + \sum_i x_i \partial x_i) + 2m^2\partial_{m^2} + (4-d)u\partial_u\right] G^{(n)}(x_1,\ldots,x_n) = 0 \tag{1.159}$$

Noting that $m^2\partial_m 2 = \frac{1}{2} m\partial_m$, we get

$$\left[\left(\sum_i x_i \partial_{x_i}\right) - m\partial_m - (4-d)u\partial_u\right] G^{(n)}(\{x_i\}) = -nD_\phi G^{(n)}(\{x_i\}) \tag{1.160}$$

eqn:bareWardG

This is the key result, the Ward identity that takes into account scale invariance breaking terms.

Lets get a hang of this. What eqn. (1.160) is that

$$G^{(n)}(\{sx_i\}, s^{-1}m, s^{-(4-d)}u) = s^{-nD_\phi} G^{(n)}(\{x_i\}, m, u) \tag{1.161}$$

Set $u = 0$, to see the physics in a free massive system. We see that, choosing $s = m$

$$G^{(n)}(\{x_i\}, m) = m^{nD_\phi} G^{(n)}(\{mx_i\}, 1) = m^{nD_\phi} F(\{mx_i\}) \tag{1.162}$$

which plainly put is the hardest way yet to do dimensional analysis. Lets record some more useful results.

$$\left[(\sum_i k_i \partial_{k_i}) + m\partial_m + (4-d)u\partial_u\right] G^{(n)}(\{k_i\}) = n(D_\phi - d) G^{(n)}(\{k_i\}) \tag{1.163}$$

and

$$\left[(\sum_i k_i \partial_{k_i}) + m\partial_m + (4-d)u\partial_u\right]\bar{\Gamma}^{(n)}(\{k_i\}) = (-nD_\phi + d)\,\bar{\Gamma}^{(n)}(\{k_i\})$$

eqn:bareWard (1.164)

This final formula is our workhorse, the Ward identity for the vertex function.

For a scale invariant system, one expects eqn. (1.164) to be an exact result. The key issue, however, is that it is unclear how to interpret this equation given that $\bar{\Gamma}^{(n)}$ in the perturbation theory are divergent quantities as seen, for example, in eqn. (1.129).

The idea now is to define a *renormalized* theory, that produces finite vertices. Generically, this entails the following. All coupling constants and fields in eqn. (1.126) are treated as *bare* quantities and will be said denoted by a subscript $\circ$, i e., the bare field $\boldsymbol{\phi}_\circ$, bare mass $m_\circ$ and the bare coupling constant $u_\circ$, with

$$h_\circ[\boldsymbol{\phi}_\circ] = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\phi}_\circ)^2 + \frac{1}{2}m^2{}_\circ\boldsymbol{\phi}_\circ{}^2 + \frac{u_\circ}{2}(\boldsymbol{\phi}_\circ{}^2)^2$$

eqn:HamBare (1.165)

All quantities calculated using this theory, for example $\bar{G}_\circ^{(n)}$ and $\bar{\Gamma}_\circ^{(n)}(\{k_i\})$ diverge, in general.

We now regularize all the vertices (at all orders in perturbation theory) by following any of the regularization techniques. For example, we could introduce an ultraviolet cutoff Λ, or as we will do later, egularize by dimensional regularization by introducing a mass scale μ. For now, lets work with the cut off scale. The idea now is to define a set of renormalized quantities (all quantities without subsript)

$$\begin{aligned}\boldsymbol{\phi}_\circ(x) &= \sqrt{Z_\phi}\boldsymbol{\phi}(x)\\ m_\circ{}^2 &= \frac{Z_{m^2}}{Z_\phi}m^2\\ u_\circ &= \frac{Z_u}{Z_\phi^2}u\end{aligned}$$

eqn:Phi4RC (1.166)

such that the parameters m_o and u_o are "divergent" and that the quantities m^2 and u the renormalized mass and coupling constant are finite. This means that the renormalization parameters must be divergent and may depend on stuff like Λ (or ϵ or μ in case of DR). We immediately have that

$$\bar{\Gamma}_\circ^{(n)}(\{k_i\}) = Z_\phi^{-n/2}\bar{\Gamma}^{(n)}(\{k_i\}; m, u)$$

eqn:RenGam (1.167)

Now it strikes! The Ward identity eqn. (1.164) cannot "be correct" as there the ultraviolet scale hidden in plain sight! In other words, the Ward identity contains an anomaly. For example, with the cutoff regularization, we have to have

$$\left[(\sum_i k_i\partial_{k_i}) + m_\circ\partial_{m_\circ} + (4-d)u_\circ\partial_{u_\circ} + \Lambda\partial_\Lambda\right]\bar{\Gamma}^{(n)}_\circ(\{k_i\}) = (-nD_\phi + d)\,\bar{\Gamma}^{(n)}_\circ(\{k_i\}) \tag{1.168}$$

eqn:bareWardAnom

However, the physical vertex will (should) not have an anomaly! How is this rescued? What happens is that mass dimension of the field changes from D_ϕ to Δ_ϕ such that

$$\left[(\sum_i k_i\partial_{k_i}) + m\partial_m + (4-d)u\partial_u\right]\bar{\Gamma}^{(n)}(\{k_i\}) = (-n\Delta_\phi + d)\,\bar{\Gamma}^{(n)}(\{k_i\}) \tag{1.169}$$

In other words the apparent anomaly in the Ward identity is "cured" by the field picking up an anomalous dimension even while keeping all quantities finite.

This is the overall idea.

Now the key point is that the Zs are function of the renormalized parameters m, u and Λ (μ and ϵ in case of DR). Here is what happens, as $m \to 0$, we find that the renormalized dimensionless coupling constant g goes to a finite value $g^\star$, and Zs start to show powerlaw behaviour. For example, in this limit,

$$\begin{aligned}
\bar{\Gamma}^{(2)}_\circ(k) &\approx \left(\frac{\Lambda}{m}\right)^{\eta}\bar{\Gamma}^{(2)}(k)\\
{m_\circ}^2 &\approx \left(\frac{\Lambda}{m}\right)^{\Delta_{m^2}} m^2\\
u_\circ &\approx \left(\frac{\Lambda}{m}\right)^{\eta_u+(4-d)} u
\end{aligned} \tag{1.170}$$

where the numbers $\eta, \Delta_{m^2}, \eta_u$ are determined by the dimensionless constant $g^\star$. This is how universal behaviour (independent of ultraviolet details) emerges.

Time to get the big picture. Idea is that any field theory has infinities. The question is can we "handle" the infinities. It is possible in certain field theories called as renormalizable theories. In these theories the infinities can be traced just a few qauntum processes (so called *primitive divergences*), and in fact, infinities that occur in any quantum process can be traced to these primitive divergences. At any order of perturbation theory, we can get rid of these primitive divergences by defining a renormalized field theory redefining the field and some coupling constants (for the ϕ^4, these are shown in eqn. (1.166)). These then allow us to see how anomalous dimensions can emerge out of eqn. (1.168).

Suppose we go through this process, how can we obtain information, for example, the critical exponents? We will discuss the final outcome in terms of the dimensional regularized minimal subtraction (MS) scheme of 'tHooft and Veltman. Do not yet worry about what MS actually means, we do not need to understand that now. The central notion is that the bare (divergent) vertex functions $\bar{\Gamma}^n_\circ$ depend on $m_\circ$, $u_\circ$ and $\epsilon = 4 - d$. The finite renormalized functions, on the other hand, depend on renormalized parameters m, u, ϵ and, *most importantly on the renormalization scale* μ. We express the renormalized interaction via $\mu = g\mu^\epsilon$ here g is the dimensionless renormalized coupling constant. The state of affairs is summarized in the equation, which is an explicit restatement of eqn. (1.171)) for the MS scheme

eqn:RenGam
$$\bar{\Gamma}^{(n)}_\circ(\{k_i\}; m_\circ, u_\circ, \epsilon) = Z_\phi^{-n/2} \bar{\Gamma}^{(n)}(\{k_i\}; m, g, \mu, \epsilon) \tag{1.171}$$

In general, Z_s, such as those defined in eqn. (1.166), depend on all on m, g, μ, ϵ. Suppose we change μ, what happens? Well, first, we observe that the bare quantities should not change! This means that the right hand side of eqn. (1.171) must satisfy the following differential equation (for a fixed ϵ)

$$\left[\mu\partial_\mu + (\mu\partial_\mu m)\partial_m + (\mu\partial_\mu g)\partial_g - n(\mu\partial_\mu \log(\sqrt{Z_\phi}))\right] \bar{\Gamma}^{(n)}(\{k_i\}; m, g, \mu, \epsilon) = 0 \tag{1.172}$$

called the *renormalization group equation*. It is conventional to define the *renormalization group flow equations* as

eqn:Zbeta
$$\gamma(m, g, \mu, \epsilon) = \mu\partial_\mu \log(\sqrt{Z_\phi}) \tag{1.173}$$

eqn:mbeta
$$\gamma_m(m, g, \mu, \epsilon) = \frac{1}{m}\mu\partial_\mu m \tag{1.174}$$

eqn:gbeta
$$\beta(m, g, \mu, \epsilon) = \mu\partial_\mu g \tag{1.175}$$

so that maximum confusion can be created with the critical exponents! No the γ and β here are NOT the critical exponents γ and β. These are very similar in spirit to Wilson's idea of what happens when a scale is changed, for example suppose we define $\mu(s) = \mu s^{-1}$, then all $\mu\partial_\mu \to -s\partial_s$, and we get the RG flow equations! We see that this the program of renormalized field theory is simply a different way of doing the "integrate/rescale/renormalize" procedure of Wilson all in one shot. Once we have the RG equations, we are in business. We can find fixed points and obtain the critical exponents by linearizing the RG equations about the interesting fixed point. Fantastic, isn't it?

Actually, MS is even more fantastic. Quite amazingly, we will find that the Zs are independent of m and μ, but depend only on g and ϵ! Who says miracles are not possible? In the MS scheme, the RG flow equations simplify quite a bit in that the lhs quantities in eqn. (1.173), eqn. (1.174) and eqn. (1.175) are functions of g and ϵ alone, leading to

$$\left[\mu\partial_\mu + \gamma_m(g,\epsilon)m\partial_m + \beta(g,\epsilon)\partial_g - n\gamma(g,\epsilon)\right)]\,\bar{\Gamma}^{(n)}(\{k_i\};m,g,\mu,\epsilon) = 0 \quad \text{eqn:MSRG} \tag{1.176}$$

Further, we see that fixed point equation

$$\beta(g,\epsilon) = 0 \tag{1.177}$$

will lead to a dimensionless fixed point coupling $g^\star$. It is quite clear now that everything near $g^\star$ is determined by value of $g^\star$, and in particular, the critical exponents! One sees how the universality arises fairly easily in the MS scheme.

It is also good to get a slightly different perspective. Eqns. eqn. (1.173), eqn. (1.174) and eqn. (1.175), and be interpreted differently, as done by field theory folks. They describe the scale μ dependence of coupling constants, i. e., *flowing coupling constants* as they say. For example if $g \to 0$ as $\mu \to \infty$, the theory would be called asymptotically free!

We have pushed as far as we can with our philosophy. Now we need to get down to doing some calculations to illustrate these ideas, and elaborate on the advertised fantastic properties of the MS scheme. Before we get down to the calculations, let us get a general overview of *when* a renormalization procedure such as that envisaged in eqn. (1.166) is possible. For this purpose, we consider a more general theory where the ϕ^4 interaction is replaced by a ϕ^p interaction. We start by asking if an arbitrary diagram F contributing to $\bar{\Gamma}_\circ^{(n)}$ has divergences. Clearly such a diagram will have n external legs. Let us suppose that there are v interaction vertices. Then the number of internal lines I is given by

$$I = \frac{1}{2}(vp - n). \tag{1.178}$$

Leading to a total number of $I+n$ lines, i. e., $I+n$ momenta. How many of these are independent? Well the n external lines already have independent momenta with (one constraint of total momentum conservation). For the v vertices, we have $v-1$ independent momentum constrants (the -1 comes from total momentum conservatoin of external lines). Thus the number independent momenta, i. e., *loops* of the diagram is equal to

$$L = (I+n) - (n+v-1) = I - (v-1). \tag{1.179}$$

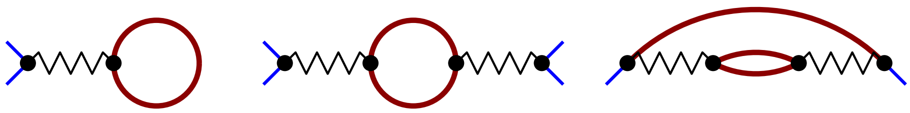
One immediately sees that these independent momenta will be integrated over to get the value of the diagram. Taking the usual form of the kinetic energy and mass term in d dimensions we obtain the mass dimension of the diagram F (only the integration part, without the strength of the interaction vertices) as

$$\Delta(F) = dL - 2I = (d-2)\left[\frac{(pv-n)}{2}\right] - d(v-1) \tag{1.180}$$

A diagram with $\Delta(F) > 0$ is called "divergent", while if $\Delta(F) < 0$, it is scalled "superficially convergent". If $\Delta(F) = 0$ then there is a log behaviour ("marginal"). A superficially convergent digram can have *subdivergences*. If the theory has only a few diagrams that are truly "divergent", one can construct a renormalizable theory. ✍ **Elaborate on this discussing subdivergences.** For ϕ^4 theory in $d=4$, one can show that

$$\Delta(F) = 4 - n \tag{1.181}$$

and that there are only three diagrams that are truly divergent. In other words, if we "take care" of these, we are done. These diagrams are



Tadpole ($\boldsymbol{T}$) Bubble ($\boldsymbol{B}$) Sunrise ($\boldsymbol{S}$)

and if fact it can be shown that *all* divergences in $\phi^4/d=4$ arise from these diagrams.

The determination of Zs proceeds via a *renormalized perturbation theory*. One formally starts with

$$h[\phi] = \frac{1}{2}(\nabla\boldsymbol{\phi})^2 + \frac{1}{2}m^2\boldsymbol{\phi}^2 + \frac{g\mu^\epsilon}{4!}(\boldsymbol{\phi}^2)^2 + \frac{Q_\phi}{2}(\nabla\boldsymbol{\phi})^2 + \frac{Q_m m^2}{2}\boldsymbol{\phi}^2 + \frac{Q_g g\mu^\epsilon}{4!}(\boldsymbol{\phi}^2)^2 \tag{1.182}$$

eqnRenHam

where the terms with Q are called the *counterterms*, which are calculated at any order of perturbation theory to keep all $\bar{\Gamma}^{(n)}$ finite *up to that order of perturbation*. Evidently, as one goes to higher order in perturbation theory, Qs will accumulate higher powers of g.

A close look shows that this is simply a recasting of the bare theory. For, define

$$\begin{aligned} Z_\phi &= (1 + Q_\phi) \\ Z_m &= (1 + Q_m) \\ Z_g &= (1 + Q_g) \end{aligned} \tag{1.183}$$

and after some minor algebra one sees that $h[\boldsymbol{\phi}] = h_\circ[\boldsymbol{\phi}_\circ]$.

Now on to the calculation of Qs. At one loop level, we need to consider only the tadpole and the bubble. After some damage to the rain forest, we find

$$\boldsymbol{T} = m^2 \frac{g}{(4\pi)^2}\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2}{m^2}\right) + \psi(2) + O(\epsilon)\right] \tag{1.184}$$

$$\boldsymbol{B} = u\frac{g}{(4\pi)^2}\left[\frac{2}{\epsilon} + \int_0^1 \mathrm{d}t \log\left(\frac{4\pi\mu^2}{m^2 + t(1-t)q^2}\right) + \psi(1) + O(\epsilon)\right] \tag{1.185}$$

(q in $\boldsymbol{B}$ is the "momentum flowing" through the bubble; what happens at $q = 0$ is the key object of interest for us). Now we introduce the *minimal subtraction scheme* to calculate the counterterms. This involves, removing only the ϵ pole terms after taking in the contribution of $\boldsymbol{T}$ and $\boldsymbol{B}$ (this involves multiplying $\boldsymbol{T}$ and $\boldsymbol{B}$ by appropriate factors including topological and expansion factorials) to $\bar{\Gamma}^{(2)}$ and $\bar{\Gamma}^{(4)}$ respectively. Some algebra gives (result quoted are for $N = 1$)

$$Z_\phi^{(1-\text{loop})} = 1 \tag{1.186}$$

$$Z_m^{(1-\text{loop})} = 1 + \frac{g}{(4\pi)^2}\frac{1}{\epsilon} \tag{1.187}$$

$$Z_g^{(1-\text{loop})} = 1 + \frac{3g}{(4\pi)^2}\frac{1}{\epsilon} \tag{1.188}$$

Even at the one loop level, we see some really nice things. For example, the β function is

$$\mu\partial_\mu g|_{(1-\text{loop})} = -\epsilon g + \frac{3g^2}{(4\pi)^2} \tag{1.189}$$

eqn:Beta1loop

from which we immediately obtain the non-trivial Wilson-Fisher fixed point $g^\star = \frac{(4\pi)^2}{3}\epsilon$. ✍ **Discuss two loop focussing on 0) How subdivergences give to higher poles of ϵ 1) S, overlapping divergences giving "log" terms, nontrival Q_ϕ** I will quote here the results of the two loop calculations (please note that I have not obtained all terms of this by myself, I am quoting from the literature)

$$Z_\phi^{(2-\text{loop})} = 1 - \frac{1}{12}\left(\frac{g}{(4\pi)^2}\right)^2 \frac{1}{\epsilon} \tag{1.190}$$

$$Z_m^{(2-\text{loop})} = 1 + \left[\frac{g}{(4\pi)^2} - \frac{1}{2}\left(\frac{g}{(4\pi)^2}\right)^2\right]\frac{1}{\epsilon} + 2\left(\frac{g}{(4\pi)^2}\right)^2 \frac{1}{\epsilon^2} \tag{1.191}$$

$$Z_g^{(2-\text{loop})} = 1 + 3\left[\frac{g}{(4\pi)^2} - \left(\frac{g}{(4\pi)^2}\right)^2\right]\frac{1}{\epsilon} + 9\left(\frac{g}{(4\pi)^2}\right)^2 \frac{1}{\epsilon^2} \tag{1.192}$$

We can now summarize the results of the renormalized perturbation theory, that redeems the promise of the magic of the MS scheme made near eqn. (1.176). Suppose we calculate up to L loops, then we find,

$$Z_\phi = 1 + \sum_{l=1}^{L(\infty)} \frac{A_{\phi,l}(g)}{\epsilon^l} \tag{1.193}$$

eqn:ZphiSeries

$$Z_m = 1 + \sum_{l=1}^{L(\infty)} \frac{A_{m,l}(g)}{\epsilon^l} \tag{1.194}$$

eqn:ZmSeries

$$Z_g = 1 + \sum_{l=1}^{L(\infty)} \frac{A_{g,l}(g)}{\epsilon^l} \tag{1.195}$$

eqn:ZgSeries

where all A_ls is a *polynomial* in g of order L, as is immediately evident from our one loop and two loop results. Indeed, if we could calculate to *all orders*[4] as indicated by ∞ in brackets, and the A_ls then are power series.

We are now in more happy territory of generalities! Once we have the Zs, we can calculate the flow equations. The MS magic, provides great simplification here, because in MS, Zs depends only on g and ϵ leading to

$$\gamma(g,\epsilon) = \frac{\beta(g,\epsilon)}{2}\frac{Z'_\phi}{Z_\phi} \tag{1.196}$$

eqn:phiZp

$$\gamma_m(g,\epsilon) = -\frac{\beta(g,\epsilon)}{2}\frac{Z'_m}{Z_m} + \gamma(g,\epsilon) \tag{1.197}$$

eqn:mZp

$$((gZ_g)'Z_\phi - 2(gZ_g)Z'_\phi)\beta(g,\epsilon) = -\epsilon(gZ_g)Z_\phi \tag{1.198}$$

eqn:gZp

where $()' = \frac{\mathrm{d}}{\mathrm{d}g}$. The last equation eqn. (1.198) proves to be a crack in a dark room, which we can pry open! Noting that eqn. (1.193) etc., have only

[4] Assuming an infinite supply of *live* graduate students.

inverse powers of ϵ, we are able to infer that $\beta(g,\epsilon)$ must be a *linear* polynomial in ε! In an equation, $\beta(g,\epsilon) = b_0(g) + b_1(g)\epsilon$, and can be explicitly computed as

eqn:betagfin

$$\beta(g,\epsilon) = \underbrace{-g}_{b_1(g)}\,\epsilon + \underbrace{g^2(A'_{g,1} - 2A'_{\phi,1})}_{b_0(g)}. \tag{1.199}$$

We can futher obtain,

eqn:betaphifin

$$\gamma(g,\epsilon) = -\frac{1}{2} g A'_{\phi,1} \tag{1.200}$$

eqn:betamfin

$$\gamma_m(g,\epsilon) = \frac{1}{2} g \left(A'_{m,1} - A'_{\phi,1}\right) \tag{1.201}$$

This analysis also puts many constraints on A_ls. The truly breathtaking and remarkable result is that only the terms associated with the simple poles of Zs appear in the RG flow equations eqn. (1.199), eqn. (1.200) and eqn. (**??**), and they do not have any singular behaviour when $\epsilon \to 0$. This is it! We now know how our system "changes" as we change scales. By imposing the vanishing of the β-function, we can find the fixed points and study the physics close to them. Let us do just that.

To be with condensed matter folks, we need to run towards the infrared. For this purpose, identify μ to be a base scale and write

$$\mu(s) = s^{-1}\mu \tag{1.202}$$

which is the the "inverse" version of the eqn. (1.27). We this find, writing $s = e^{\ell}$

eqn:gWilson

$$s\partial_s g = \frac{\mathrm{d}g}{\mathrm{d}\ell} = -\beta(g) \tag{1.203}$$

eqn:m2Wilson

$$s\partial_s \bar{m}^2 = \frac{\mathrm{d}\bar{m}^2}{\mathrm{d}\ell} = 2(1-\gamma_m(g))\bar{m}^2 \tag{1.204}$$

where we have, following our Wilsonian infrared pursuit, defined $m = \bar{m}\mu$, $\bar{m}$ is the dimensionless mass. Dependence of β and γ on ϵ will not be explicitly shown here are hence forth. Solving the two equations, we get

eqn:gsol

$$s = e^{-\int_{g(1)}^{g(w)} \mathrm{d}w \frac{1}{\beta(g(w))}} \tag{1.205}$$

which is used to obtain $g(s)$. Once we have this we can obtain

eqn:msol

$$\bar{m}^2(s) = \bar{m}^2(1) e^{\int_1^s \frac{\mathrm{d}w}{w} 2[1-\gamma_m(g(w))]}. \tag{1.206}$$

We can also solve eqn. (1.176) (by noting that $\mu\partial_\mu + \gamma_m m\partial_m + \beta\partial_g \equiv -s\frac{\mathrm{d}}{\mathrm{d}s}$) as

$$\bar{\Gamma}^{(n)}\left(\{k_i\}; m(s), g(s), \mu s^{-1}\right) = \left[e^{-n\int_1^s \frac{\mathrm{d}w}{w}\gamma(g(w))}\right]\bar{\Gamma}^{(n)}\left(\{k_i\}; m(1), g(1), \mu\right) \quad (1.207)$$

eqn:Gamsol

These last three equations above have everything we need to obtain the critical physics.

Lets first obtain the fixed points. Set the rhs of eqn. (1.203) and eqn. (1.204) to zero, to obtain

$$\beta(g^\star) = 0 \quad (1.208)$$
$$\bar{m}^\star = 0 \quad (1.209)$$

At the critical point γ and γ_m take on values $\gamma^\star$ and $\gamma_m^\star$ calculated by sticking $g^\star$ into eqns. (1.200) and (1.201). It is clear that there are two fixed points: the Gaussian fixed point, and the Wilson-Fisher fixed point. At the Gaussian fixed point

$$g_G^\star = 0, \bar{m}_G^{2\star} = 0, \gamma_G^\star = 0, \gamma_{mG}^\star = 0 \quad (1.210)$$

On the other hand at the Wilson-Fisher fixed point

$$g_{WF}^\star = \frac{(4\pi)^2}{3}\epsilon + ..., \bar{m}_{WF}^{2\star} = 0, \gamma_{WF}^\star \sim \epsilon^2, \gamma_{mWF}^\star \sim \epsilon \quad (1.211)$$

We will only dicuss the WF fixed point, and hence drop all the WF suffixes. Suppose we are close the the critical point, i e., start with $g = g^\star + \delta g$ and a small $\bar{m}^2 \neq 0$, then the flow will take it to

$$\begin{aligned}\delta g(s) &= \delta g s^{-\omega^\star} \\ \bar{m}^2(s) &= \bar{m}^2 s^{2(1-\gamma_m^\star)}\end{aligned} \quad (1.212)$$

where

$$\omega^\star = \left.\frac{\mathrm{d}\beta}{\mathrm{d}g}\right|_{g=g^\star} \quad (1.213)$$

For the WF fixed point, to one loop order, $\omega^\star \approx \epsilon$. This means δg flows to zero under RG, where as $\bar{m}$ grows.

To connect to the Landauesque language to obtain critical exponents etc., we need a parameter analogous to t as defined in eqn. (1.10). Given that what we have control on in our calculation is μ, or in other words, s, we have to hunt down what to call t. We start this pursuit by noting that we should be close to a fixed point for t to have a meaning. That $\bar{m}^{2\star} = 0$

makes a lot of sense as we expect the system to be gapless. Yes, this is true, but m^2 is not the true gap in the system, but a quantity that scales monotonically with the true gap, i. e., when $\bar{m}^2 \to 0$ the gap in the system also goes to zero. Suppose, we start close to the critical point; then it is natural to call

$$\bar{m}^2(s=1) = t \tag{1.214}$$

In this set up, when is $t=1$? It natural to ascribe $t=1$ when the m scale is μ, i.e., when $\bar{m}=1$. At what value of s, call it s_m, does this happen? From eqn. (1.206), and assuming being very close to the critical point, we get

eqn:sm

$$1 = t s_m^{2(1-\gamma_m^\star)} \implies s_m = t^{-\frac{1}{2(1-\gamma_m^\star)}} \tag{1.215}$$

Now we will use properties of vertex function to obtain the how the inverse correlation length scales with t. First recall eqn. (1.146). It is evident from dimensional analysis that (note that we have changed m to $\bar{m}$, no issue there)

eqn:LamScale

$$\bar{\Gamma}^n(\{k_i\},\bar{m},g,\mu) = \lambda^{(nD_\phi-d)}\bar{\Gamma}^n(\{\lambda k_i\},\bar{m},g,\lambda\mu) \tag{1.216}$$

for any scale factor λ. Noting that for $t\to 0$, $s_m\to\infty$ from eqn. (1.215), and using eqn. (1.207), we get

$$\bar{\Gamma}^{(n)}\left(\{k_i\};\bar{m}(1),g(1),\mu\right) = s_m^{n\gamma^\star}\bar{\Gamma}^{(n)}\left(\{k_i\};\bar{m}(s_m),g(s_m),\mu s_m^{-1}\right) \tag{1.217}$$

We now use eqn. (1.216) with $\lambda = \mu^{-1}s_m$ in the rhs of the above equation to obtain

$$\bar{\Gamma}^{(n)}\left(\{k_i\};\bar{m}(1),g(1),\mu\right) = \mu^{-(nD_\phi-d)}s_m^{(nD_\phi-d)+n\gamma^\star}\bar{\Gamma}^{(n)}\left(\{\frac{s_m k_i}{\mu}\};1,g^\star,1\right) \tag{1.218}$$

Looking at $n=2$, we find

$$\bar{\Gamma}^{(2)}\left(\{k\};\bar{m}(1),g(1),\mu\right) = \mu^2 s_m^{-(2-2\gamma^\star)}f(\frac{s_m k}{\mu}) \tag{1.219}$$

leading immediately to the correlation length

eqn:Gam2Sol

$$\xi(t) = \mu^{-1}s_m = \mu\, t^{-\frac{1}{2(1-\gamma_m^\star)}} \tag{1.220}$$

immediately providing the critical exponent

$$\nu = \frac{1}{2(1-\gamma_m^\star)} \tag{1.221}$$

Now what can we say about the dimensionless function $f(s_m k)$? Well, as $t \to 0$, $s_m \to \infty$. If $\bar{\Gamma}^{(2)}$ to be nontrivial (which it must be), we must have that

$$f(x) \sim x^{2-2\gamma^\star}, \tag{1.222}$$

leading to critical two point vertex as

$$\bar{\Gamma}^{(2)\star}(k) \sim \mu^{2\gamma^\star}|k|^{2-2\gamma^{\star}}` \tag{1.223}$$

which gives

$$G^{(2)\star}(x) \sim \frac{1}{|x|^{d-2+2\gamma^\star}} \tag{1.224}$$

resulting in another result for a the critical exponent

$$\eta = 2\gamma^\star \tag{1.225}$$

Finally we see the emergence of the anomalous dimension (ϵ^2 at the 2-loop level)at the critical point!

2

Nonlinear σ Models

A very useful model for the field theoretical study of many problems in condensed matter physics is the nonlinear σ model (NLσM). NLσMs have proved useful in the study of magnetism, disordered systems, and even in the *definition* of topological phases (more about this later). The simplest avatar of this model is the, so called $O(N)$ NLσM, where the field is an N component unit vector, i. e., the field at every point on the arena lives on the surface of a sphere. The notion of "sphere" can be generalized to any symmetric space of Cartan and indeed this is what makes them useful in the description of disordered fermions and in the characterization of topological insulators.

Much of what we will discuss is inspired by PolyakovBook[**?**], and a seminal paper by Brezin-Zinn-JustinPRB1976[**?**].

Consider a d-cubic lattice with sites labelled by i (lattice spacing a) which host N-component vectors $\boldsymbol{n}_i$ at each site. The vector $\boldsymbol{n}_i$ satisfies

eqn:Constraint

$$\boldsymbol{n}_i \cdot \boldsymbol{n}_i = 1 \tag{2.1}$$

One can write a model (Heisenberg ferromagnet) that has a global $O(N)$ symmetry as

$$H = -J \sum_{\langle ij \rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j \tag{2.2}$$

The partition function at temperature T can be written as

$$\mathcal{Z} = \int \prod_i \mathrm{d}\boldsymbol{n}_i \prod_i \delta(\boldsymbol{n}_i \cdot \boldsymbol{n}_i - 1) e^{\frac{J}{T} \sum_{\langle ij \rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j} \tag{2.3}$$

One can see the ground state will prefer to have all $\boldsymbol{n}_i$ equal, and this state breaks the $O(N)$ symmetry. On the other hand at $T \to \infty$, any configuration is equally likely and this state is $O(N)$ symmetric. The question is if there is phase transition in going from $T=0$ to $T=\infty$.

Our next step is to make this into a "field theory". We will do this by staying at low temperatures. we write

eqn:pidefn

$$\boldsymbol{n}_i \equiv (n_{i1}, n_{i2}, \ldots, n_{iN}) \equiv (\sigma_i, \boldsymbol{\pi}_i) \tag{2.4}$$

where $\boldsymbol{\pi}_i$ is a vector with $(N-1)$ components. Obviously,

eqn:constr

$$\sigma_i^2 + \boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_i = 1 \tag{2.5}$$

We can now use this last condition to eliminate σ_i leadding to

$$\mathcal{Z} = \int \prod_i \mathrm{d}\boldsymbol{\pi}_i e^{\frac{J}{T}\sum_{\langle ij\rangle} \boldsymbol{n}_i\cdot\boldsymbol{n}_j - \frac{1}{2}\ln(1-\boldsymbol{\pi}_i\cdot\boldsymbol{\pi}_i)} \tag{2.6}$$

Now use the identity

$$\boldsymbol{n}_i \cdot \boldsymbol{n}_j = -\frac{1}{2}(\boldsymbol{n}_j - \boldsymbol{n}_i)^2 + 1 \tag{2.7}$$

and dump the constant. One then gets the following field theory

$$\mathcal{Z} = \int D[\boldsymbol{\pi}] e^{-S[\boldsymbol{\pi}]} \tag{2.8}$$

where

eqn:Spi

$$\begin{aligned} S[\boldsymbol{\pi}] &= \frac{1}{2g_\circ}\int \mathrm{d}^d x (\boldsymbol{\nabla}\boldsymbol{n})^2 - \frac{a^{-d}}{2}\int \mathrm{d}^d x \ln(1-\boldsymbol{\pi}^2) \\ &= \frac{1}{2g_\circ}\int \mathrm{d}^d x \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + (\boldsymbol{\nabla}\sqrt{1-\boldsymbol{\pi}^2})^2\right] + \frac{a^{-d}}{2}\int \mathrm{d}^d x \ln(1-\boldsymbol{\pi}^2) \end{aligned} \tag{2.9}$$

Here

$$g_\circ = \frac{T}{J}a^{d-2}. \tag{2.10}$$

It is standard to express

$$S[\boldsymbol{n}] = \frac{1}{2g_0}\int \mathrm{d}^d x (\boldsymbol{\nabla}\boldsymbol{n})^2 \tag{2.11}$$

with the constraint of unit $\boldsymbol{n}$ being understood.

First let us introduce a momentum scale μ. We see that

$$[\![g_\circ]\!] = \mu^{2-d} \tag{2.12}$$

Thus the coupling constant $g_\circ$ is dimensionless in $d = 2$.

Anticipating some future developments, we introduce a bare external field $b_\circ$ which couples to $\sigma_\circ$ and write the bare action

$$S_\circ[\boldsymbol{\pi}_\circ] = \frac{1}{2g_\circ}\int \mathrm{d}^d x \left[(\boldsymbol{\nabla}\boldsymbol{\pi}_\circ)^2 + (\boldsymbol{\nabla}\sqrt{1-\boldsymbol{\pi}_\circ{}^2})^2\right] + \frac{a^{-d}}{2}\int \mathrm{d}^d x \ln(1-\boldsymbol{\pi}_\circ{}^2)$$
$$- \frac{2b_\circ}{2g_\circ}\int \mathrm{d}^d x \sqrt{1-\boldsymbol{\pi}_\circ{}^2} \qquad (2.13)$$

eqn:NLSbare

We see that

$$[\![b_\circ]\!] = \mu^2 \qquad (2.14)$$

That this theory is renormalizable is very interesting and curious. ✍ **Discuss renormalizability etc.** Now we introduce renormalized fields

$$(\sigma_\circ, \boldsymbol{\pi}_\circ) = (\sqrt{Z_n}\sigma, \sqrt{Z_n}\boldsymbol{\pi}) \qquad (2.15)$$

Note that the renormalized fields satisfy,

$$\sigma^2 + \boldsymbol{\pi}^2 = \frac{1}{Z_n} \qquad (2.16)$$

leading to

$$\sqrt{Z_n}\sigma = \sqrt{1 - Z_n\boldsymbol{\pi}^2} \qquad (2.17)$$

We now write out the renormalized action

$$S[\boldsymbol{\pi}] = \frac{1}{2Z_g g}\int \mathrm{d}^d x \left[Z_n(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + (\boldsymbol{\nabla}\sqrt{1-Z_n\boldsymbol{\pi}^2})^2\right] + \frac{a^{-d}}{2}\int \mathrm{d}^d x \ln(1-Z_n\boldsymbol{\pi}^2)$$
$$- \frac{2b}{2g}\int \mathrm{d}^d x \sigma(x) \qquad (2.18)$$

which can be massaged into the following form

$$S[\boldsymbol{\pi}] = \frac{1}{2Z_g g}\int \mathrm{d}^d x \left[\left(Z_n(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + (\boldsymbol{\nabla}\sqrt{1-Z_n\boldsymbol{\pi}^2})^2\right) - \frac{2Z_g b}{\sqrt{Z_n}}\sqrt{1-Z_n\boldsymbol{\pi}^2}\right]$$
$$+ \frac{a^{-d}}{2}\int \mathrm{d}^d x \ln(1-Z_n\boldsymbol{\pi}^2) \qquad (2.19)$$

eqn:NLSren

Here we have renormalized the coupling constant and the magnetic field

$$g_\circ = Z_g g \tag{2.20}$$

$$b_\circ = \frac{Z_g b}{\sqrt{Z_n}} \tag{2.21}$$

For small g, we note that

$$\langle \boldsymbol{\pi}^2 \rangle \sim g_\circ \tag{2.22}$$

So one might readily expand the nonlinear functions in eqn. (2.9) in powers. This is achieved by using

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \ldots, \qquad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots \tag{2.23}$$

eqn:expns

Now we perform the following tricks: First, note

$$Z_g = 1 + Q_g; Z_n = 1 + Q_n; \frac{Z_n}{Z_g} = 1 + Q \tag{2.24}$$

eqn:NLSQdef

Using the expansions eqn. (2.23), we express

$$\begin{aligned} S[\boldsymbol{\pi}] =& \frac{1}{2g}\int \mathrm{d}^d x \left[\left(\frac{Z_n}{Z_g}(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + \frac{Z_n^2}{4Z_g}(\boldsymbol{\nabla}\boldsymbol{\pi}^2)^2 \right) + b\sqrt{Z_n}\boldsymbol{\pi}^2 + b\frac{Z_n^{3/2}}{4}(\boldsymbol{\pi}^2)^2 \right] \\ &+ \frac{a^{-d}}{2}\int \mathrm{d}^d x \ln(1 - Z_n\boldsymbol{\pi}^2) \end{aligned} \tag{2.25}$$

We will use dimensional regularization to renormalize the theory. First, a bit of a shocker. The term a^{-d} *can be dropped*! This is because

$$a^{-d} \sim \int \mathrm{d}^d k = 0 \qquad ! \tag{2.26}$$

No, the "!" does not stand for factorial, it is an exclamation! To evaluate the renormalization factors, we go to k space. Now assume g is small, we expect Qs in eqn. (2.24)to be small as well, giving for example, $Q = Q_n - Q_g$ etc., to get

$$\begin{aligned} S[\boldsymbol{\pi}] =& \frac{1}{2g}\int \mathrm{d}^d x \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + b\boldsymbol{\pi}^2 + \frac{1}{4}\left((\boldsymbol{\nabla}\boldsymbol{\pi}^2)^2 + b(\boldsymbol{\pi}^2)^2 \right) \right] \\ &+ \frac{1}{2g}\int \mathrm{d}^d x \left[Q(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + \frac{Q_n}{2}b\boldsymbol{\pi}^2 + \frac{Q_n + Q}{4}(\boldsymbol{\nabla}\boldsymbol{\pi}^2) + b\frac{3Q_n}{8}(\boldsymbol{\pi}^2)^2 \right] \end{aligned} \tag{2.27}$$

The idea is the usual one of renormalized perturbation theory; Qs are selected so as to provide counterterms so that all correlators are rendered finite at for any given number of loops considered.

We have to go to momentum space for ease in the calculation. We get

$$\begin{aligned} S[\boldsymbol{\pi}] =& \frac{1}{2g}\left[\sum_k (k^2+b)\pi^i(k)\pi^i(-k) \right.\\ &\left. + \frac{1}{4V}\sum_{k,k_1,k_2}(k^2+b)\pi^i(k-k_1)\pi^i(k_1)\pi^i(-k-k_2)\pi^i(k_2)\right] \\ &+\frac{1}{2g}\left[\sum_k (Qk^2+\frac{Q_n}{2}b)\pi^i(k)\pi^i(-k)\right. \\ &\left. + \frac{1}{4V}\sum_{k,k_1,k_2}\left((Q_n+Q)k^2+\frac{3Q_n}{2}b\right)\pi^i(k-k_1)\pi^i(k_1)\pi^i(-k-k_2)\pi^i(k_2)\right] \end{aligned} \tag{2.28}$$

V is the d-volume of the system with periodic boundary conditions.

From this we can compute the one loop vertex

$$\begin{aligned} \Gamma^{(2)}(k) = &\frac{1}{g}(k^2+b) + \left[\frac{1}{8gV}\sum_{k'}\left(4(N-1)b\frac{g}{(k')^2+b} + 8((k')^2+b)\frac{g}{(k'-k)^2+b}\right)\right] \\ &+ \frac{1}{g}\left(Qk^2+\frac{Q_n}{2}b\right) \end{aligned} \tag{2.29}$$

Now we use that fact that

$$\begin{aligned} &\frac{1}{V}\sum_{k'}\left(4(N-1)b\frac{g}{(k')^2+b} + 8((k')^2+b)\right) \\ &= \frac{8g}{(2\pi)^d}\int \mathrm{d}^d k'\left(\frac{1}{2}(N-1)\frac{b}{(k')^2+b} + \frac{((k'+k)^2+b)}{(k')^2+b}\right) \\ &= \frac{8g}{(2\pi)^d}\int \mathrm{d}^d k'\left(\frac{1}{2}(N-1)\frac{b}{(k')^2+b} + \frac{k^2}{(k')^2+b}+1\right) \\ &\underbrace{=}_{\text{DR}} 8g\left(\frac{1}{2}(N-1)b+k^2\right)\frac{1}{(2\pi)^d}\int \mathrm{d}^d k'\frac{1}{(k')^2+b} \\ &\underbrace{=}_{\text{using eqn. (1.125)}} 8g\left(\frac{1}{2}(N-1)b+k^2\right)\frac{\mu^\epsilon}{4\pi}\left(\frac{b}{4\pi\mu^2}\right)^{\epsilon/2}\Gamma(-\frac{\epsilon}{2}) \end{aligned} \tag{2.30}$$

where we have introduced

$$d = 2 + \epsilon. \tag{2.31}$$

Now using eqn. (1.120), we get

$$\begin{aligned}
&\frac{1}{V}\sum_{k'}\left(4(N-1)b\frac{g}{(k')^2+b} + 8((k')^2+b)\right) \\
&= 8g\left(\frac{1}{2}(N-1)b+k^2\right)\frac{\mu^\epsilon}{4\pi}\left(1+\frac{\epsilon}{2}\ln\left(\frac{b}{4\pi\mu^2}\right)+\ldots\right)\left(-\frac{2}{\epsilon}+\psi(1)+\ldots\right) \\
&= 8g\left(\frac{1}{2}(N-1)b+k^2\right)\frac{\mu^\epsilon}{4\pi}\left(-\frac{2}{\epsilon}+\psi(1)-\ln\left(\frac{b}{4\pi\mu^2}\right)+\ldots\right)
\end{aligned} \tag{2.32}$$

With these developments, we see that

eqn:NLSrenG2

$$\begin{aligned}
\Gamma^{(2)}(k) &= \frac{1}{g}(k^2+b) + (\frac{1}{2}(N-1)b+k^2)\frac{\mu^\epsilon}{4\pi}\left(\psi(1)-\ln\left(\frac{b}{4\pi\mu^2}\right)\right) \\
&+ k^2\left(\frac{Q}{g}-\frac{\mu^\epsilon}{2\pi}\frac{1}{\epsilon}\right) + b\left(\frac{Q_n}{2g}-\frac{1}{2}(N-1)\frac{\mu^\epsilon}{2\pi}\frac{1}{\epsilon}\right)
\end{aligned} \tag{2.33}$$

We get the one loop counter terms as

$$\begin{aligned}
Q_n &= (N-1)\frac{g\mu^\epsilon}{2\pi\epsilon} \\
Q = Q_n - Q_g = \frac{g\mu^\epsilon}{2\pi}\frac{1}{\epsilon} \Longrightarrow Q_g &= (N-2)\frac{g\mu^\epsilon}{2\pi\epsilon}
\end{aligned} \tag{2.34}$$

Now define

$$g = \mathrm{g}\mu^{-\epsilon}, \tag{2.35}$$

leading to

$$Z_g = (1+(N-2)\frac{\mathrm{g}}{2\pi\epsilon}) \tag{2.36}$$

and

$$g_\circ = Z_g \mathrm{g}\mu^{-\epsilon} \tag{2.37}$$

from which we can obtain the β function.

$$g_\circ = \left(\mathrm{g} + \frac{(N-2)}{2\pi\epsilon}\mathrm{g}^2\right)\mu^{-\epsilon} \tag{2.38}$$

This leads to

$$-\epsilon\left(\mathrm{g}+\frac{(N-2)}{2\pi\epsilon}\mathrm{g}^2\right) + \mu\partial_\mu \mathrm{g}\left(1+\frac{(N-2)}{\pi\epsilon}\mathrm{g}\right) = 0 \tag{2.39}$$

from which we get

$$\mu\partial_\mu \mathrm{g} = \beta(\mathrm{g},\epsilon) = \epsilon \mathrm{g} - \frac{N-2}{2\pi}\mathrm{g}^2 \qquad \text{eqn:NLSUVflow} \tag{2.40}$$

Before we discuss the physics of this flow, we discuss a bit the reason for the introduction of the external magnetic field in the calculation. Look at eqn. (2.33); we see that the renormalized vertex diverges when $b \to 0$! This *is* physics, not an artifact. The issue here is that the integrals are also *infrared* divergent due to being close to the lower critical dimension. Introducing a magnetic field allows us to control these infrared divergences.

First of all, let us assume our true identity as condensed matter folks and write IR flows. This gives

$$\frac{\mathrm{dg}}{\mathrm{d}\ell} = -\mu\partial_\mu \mathrm{g} = -\epsilon \mathrm{g} + \frac{N-2}{2\pi}\mathrm{g}^2 \qquad \text{eqn:NLSUVflow} \tag{2.41}$$

We see that there are two ✍ **three?** fixed points. First,

$$\mathrm{g}_0 = 0 \tag{2.42}$$

This is an infrared stable fixed point corresponding to the broken symmetry ground state. Second,

$$\mathrm{g}_c = \frac{2\pi\epsilon}{N-2} \tag{2.43}$$

which is finite for $d > 2$ and $N > 2$! This is an IR unstable fixed point with

$$\frac{\mathrm{d}\delta g}{\mathrm{d}\ell} = \epsilon\delta \mathrm{g} \tag{2.44}$$

and corresponds to the critical point separating the ordered and disordered phase. Note that in $d = 2$ this fixed point is same as the g_0 fixed point. This is simply a statement of Mermin-Wagner theorem that one cannot break a continuous symmetry with short ranged interactions in $d \leq 2$.

We further remark that $N = 2$ ($O(2)$ symmetry) seems special. At one loop level, in $d = 2$, there is no flow for g! We will revisit this case in the next chapter.

Let us extract a bit more physics at g_c. Preparations first. Let at the m-point correalator

$$G^{(m)}(x_1,\ldots,x_m) = \langle n_{a1}(x_1)\ldots n_{am}(x_m)\rangle \tag{2.45}$$

where ai are the components. Note that

$$[\![G^{(m)}]\!] = \mu^0 \tag{2.46}$$

The momentum space version of this is

$$G^{(m)}(k_1, \ldots, k_m) = \langle n_{a1}(k_1) \ldots n_{am}(k_m) \rangle \tag{2.47}$$

and has dimensions

$$[\![G^m(\{k_i\})]\!] = \mu^{-md} \tag{2.48}$$

Owing to the expected translational symmetry of the states, we have

$$G^{(m)}(k_1, \ldots, k_m) = \delta(k_1 + k_2 + \ldots + k_m)\bar{G}^{(m)}(\{k_i\}) \tag{2.49}$$

with

$$[\![\bar{G}^{(m)}(\{k_i\})]\!] = \mu^{-(m-1)d} \tag{2.50}$$

It is these barred quantities that we will be analyzing.

The bare quantity is related to the renormalized quantity via

$$\bar{G}_\circ^{(m)}(\{k_i\}, g_\circ, b_\circ) = Z_n^{n/2}\bar{G}^{(m)}(\{k_i\}, \mathrm{g}, \mathrm{b}, \mu) \tag{2.51}$$

where we have used $b = \mathrm{b}\mu^2$. We can now write a RG equation

$$\mu \frac{\mathrm{d}Z_n^{m/2}}{\mathrm{d}\mu}\bar{G}^{(m)} + Z_n^{m/2}\mu\frac{\mathrm{d}\bar{G}^{(m)}}{\mathrm{d}\mu} = 0 \tag{2.52}$$

Define

$$\gamma_n = \beta(\mathrm{g})\frac{1}{2}\frac{Z'(\mathrm{g})}{Z(\mathrm{g})} \tag{2.53}$$

to get

$$-\mu\frac{\mathrm{d}\bar{G}^{(m)}}{\mathrm{d}\mu} = \gamma_n \bar{G}^{(m)} \tag{2.54}$$

Using our usual definition

$$\mu(s) = \frac{\mu}{s} \tag{2.55}$$

we get

$$s\frac{\mathrm{d}\bar{G}^{(m)}}{\mathrm{d}s} = \gamma_n \bar{G}^{(m)} \tag{2.56}$$

which can be solved as

eqn:NLSGRG

$$\bar{G}^{(m)}(\{k_i\}, \mathrm{g}(s), \mathrm{b}(s), \mu/s) = e^{m\int_1^s \frac{\mathrm{d}w}{w}\gamma_n(g(w))}\bar{G}^{(m)}(\{k_i\}, \mathrm{g}(1), \mathrm{b}(1), \mu) \tag{2.57}$$

where $\mathrm{g}(s)$ etc., can be solved using their respective β-functions.

First, let us look at the mass or inverse correlation length ξ^{-1}. Since there is but a single scale g in the problem (when $\mathrm{b} = 0$), we expect

$$\xi^{-1} = \mu\zeta(\mathrm{g}) \tag{2.58}$$

Given that for a given bare parameter, this relationship should be μ independent means that

$$\zeta + \beta(\mathrm{g})\frac{\mathrm{d}\zeta}{\mathrm{dg}} = 0 \tag{2.59}$$

We know from the scale invariance of the citical point that $\zeta(\mathrm{g}_c) = 0$. Integrating this equation about $\mathrm{g} = \mathrm{g}_c$, we get $(\delta\mathrm{g} = \mathrm{g} - \mathrm{g}_c)$

$$\frac{\mathrm{d}\zeta}{\zeta} = -\frac{\mathrm{dg}}{\beta(\mathrm{g})} = -\frac{1}{\beta'(\mathrm{g}_c)}\frac{\mathrm{d}\delta\mathrm{g}}{\delta\mathrm{g}} = \frac{1}{\epsilon}\frac{\mathrm{d}\delta\mathrm{g}}{\delta\mathrm{g}} \tag{2.60}$$

where we have used

$$\beta'(\mathrm{g}_c) = -\epsilon \tag{2.61}$$

We get that

$$\zeta \sim (\mathrm{g} - \mathrm{g}_c)^{1/\epsilon}. \tag{2.62}$$

or

$$\xi \sim \mu^{-1}(\mathrm{g} - \mathrm{g}_c)^{-1/\epsilon} \tag{2.63}$$

leading to

$$\nu = \frac{1}{\epsilon} \tag{2.64}$$

To obtain further results near g_c, we note

$$\frac{\mathrm{d}s}{s} = -\frac{\mathrm{dg}}{\beta(\mathrm{g})} = \frac{1}{\epsilon}\frac{\mathrm{d}\delta\mathrm{g}}{\delta\mathrm{g}} \tag{2.65}$$

or,

$$\delta\mathrm{g} \sim s^{\epsilon} \tag{2.66}$$

we run $s \to 0$ to reach the critical point. Further, a simple calculation gives

$$\gamma_n(\mathrm{g}_c) \approx \frac{(N-1)\epsilon}{2(N-2)} \tag{2.67}$$

Look at the order parameter which is $\langle\sigma(x)\rangle$; we can apply eqn. (2.57) with $m = 1$. Here

$$\bar{G}^{(1)}(k_i, g, b, \mu) = \bar{G}^{(1)}(g, b) \tag{2.68}$$

as $\bar{G}^{(1)}$ is dimensionless. Working at $b = 0$,

$$\bar{G}^{(m)}(\mathrm{g}(s)) = s^{\gamma_n(\mathrm{g}_c)}\bar{G}^{(m)}(\mathrm{g}(1)) \tag{2.69}$$

We can write $\mathrm{g}(s) = \mathrm{g}_c + \delta g$ which means

$$\langle\sigma\rangle(\delta g) = \bar{G}^{(m)}(\mathrm{g}(1))(|\delta g|)^{\frac{\gamma_n(\mathrm{g}_c)}{\epsilon}} \tag{2.70}$$

whence we get the critical exponent

$$\beta = \frac{N-1}{2(N-2)}. \tag{2.71}$$

Let us now look at the two point function $\bar{G}^{(2)}(k, \mathrm{g}, \mathrm{b}, \mu)$. Since

$$[\![\bar{G}^{(2)}]\!] = \mu^{-d} \tag{2.72}$$

From eqn. (2.57), we get

$$\bar{G}^{(2)}(k, \mathrm{g}(1), \mathrm{b}(1), \mu) = s^{-2\gamma_n(\mathrm{g}_c)}\bar{G}^{(2)}(k, \mathrm{g}(s), \mathrm{b}(s), \mu/s) \tag{2.73}$$

We can rewrite this (for $b = 0$) as

$$\bar{G}^{(2)}(k, \mathrm{g}(1), 0, \mu) = s^{-2\gamma_n(\mathrm{g}_c)} \left(\frac{\mu}{s}\right)^{-d} F(sk/\mu, \mathrm{g}(s)) \tag{2.74}$$

as $s \to 0$, $\mathrm{g} \to \mathrm{g}_c$ with

$$F(sk/\mu, \mathrm{g}_c) = f(\frac{sk}{\mu}) \tag{2.75}$$

Thus,

$$\bar{G}^{(2)}(k, \mathrm{g}(1), 0, \mu) = s^{-2\gamma_n(\mathrm{g}_c)} = \mu^{-d}s^{d-2\gamma_n(\mathrm{g}_c)}f(\frac{sk}{\mu}) \tag{2.76}$$

To get a finite answer,

$$f(x) = x^{-(d-2\gamma_n(\mathrm{g}_c))} \tag{2.77}$$

leading to

$$\bar{G}^{(2)}(k, \mathrm{g}(1), 0, \mu) = \frac{\mu^{-2\gamma_n(\mathrm{g}_c)}}{k^{d-2\gamma_n(\mathrm{g}_c)}} = \frac{\mu^{-2\gamma_n(\mathrm{g}_c)}}{k^{2+\epsilon-2\gamma_n(\mathrm{g}_c)}} \tag{2.78}$$

leading to

$$\eta = -\epsilon + 2\gamma_n(\mathrm{g}_c) = \frac{\epsilon}{(N-2)} \tag{2.79}$$

to one loop order.

2.1 Poor man's (Polyakov's) Approach

Although, the field theoretic method worked out in the last section gives us all that we need, it is still very instructive to see Polyakov's original approach to this problem.[1]

[1] Polyakov himself seems to attribute this trick to Berezinskii ✍ **check this**

The RG equation is derived by a Wilson method, employing some very clever tricks. The deep connections between NLσMs and other problems become evident in the process. Start with

$$S[\boldsymbol{n}] = \frac{1}{2g} \int \mathrm{d}^d x \, \partial_\alpha \boldsymbol{n} \cdot \partial_\alpha \boldsymbol{n} \tag{2.80}$$

Note that we do not use the "bare field" as there is an explicit momentum cutoff Λ already given (precisely as Wilson would want it). Here α runs over the spatial indices of the d-dimensional Eucledian space.

The idea now is the choose a smaller momentum scale $\tilde{\Lambda}$ related by a scale factor s to the original cutoff scale Λ and follow Wilson's three step procedure given near eqn. (1.42). Now, this is done in a very clever fashion. First, we identify the field configurations that are "slow", i. e., do *not* have wave vector components $> \tilde{\Lambda}$, we will call such a field $\boldsymbol{\sigma}(x)$ (do not confuse with the $\sigma(x)$ defined in the last section). In addition there is a fast piece $\boldsymbol{\pi}(x)$, which has only wavevectors between $\tilde{\Lambda}$ and Λ. The main idea is the

$$\boldsymbol{\sigma}(x) \cdot \boldsymbol{\sigma}(x) = 1 \tag{2.81}$$

i. e., $\boldsymbol{\sigma}$ is a *unit vector* everywhere. Now, $\boldsymbol{n}(x)$ is *also* a unit vector, which means

$$\boldsymbol{n}(x) = \sqrt{1 - \boldsymbol{\pi} \cdot \boldsymbol{\pi}} \, \boldsymbol{\sigma}(x) + \boldsymbol{\pi}(x) \tag{2.82}$$

We shall denote $\boldsymbol{\pi} \cdot \boldsymbol{\pi} = \pi^2$. Now, define a local frame (basis in the $\boldsymbol{n}$ space) such that

$$\boldsymbol{e}_0(x) = \boldsymbol{\sigma}(x), \quad \boldsymbol{e}_a, a = 1, \ldots, N-1 \tag{2.83}$$

such that

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_0 = 0; \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab} \tag{2.84}$$

We shall use the summation convention on the latin indices like a and b, and also on the spatial indices like α. We now need $\partial_\alpha \boldsymbol{n}$. To obtain this in a nice fashion, we do some ground work. First, notice

eqn:NLSM:doue0

$$\boldsymbol{e}_0(x) \cdot \boldsymbol{e}_0(x) = 1 \Longrightarrow \partial_\alpha \boldsymbol{e}_0 = A_\alpha^{a0} \boldsymbol{e}_a \equiv \boldsymbol{A}_\alpha^0 \tag{2.85}$$

Also,

$$\boldsymbol{e}_0(x) \cdot \boldsymbol{e}_a(x) = 0 \Longrightarrow \boldsymbol{e}_0 \cdot \partial_\alpha \boldsymbol{e}_a = -\boldsymbol{A}_\alpha^0 \cdot \boldsymbol{e}_a \tag{2.86}$$

Define,

$$\boldsymbol{e}_b \cdot \partial_\alpha \boldsymbol{e}_a = A_\alpha^{ab} \tag{2.87}$$

The relation $\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$ implies

$$A_\alpha^{ba} = -A_\alpha^{ab} \tag{2.88}$$

With these results we see that

$$\partial_\alpha \boldsymbol{\pi} = (D_\alpha \pi_a)\boldsymbol{e}_a - (\boldsymbol{\pi} \cdot \boldsymbol{A}^0_\alpha)\boldsymbol{\sigma} \tag{2.89}$$

where

$$D_\alpha \pi_a = \partial_\alpha \pi_a + A^{ab}_\alpha \pi_b \tag{2.90}$$

We thus get that

$$\partial_\alpha \boldsymbol{n} = \left(\partial_\alpha \sqrt{1-\pi^2} - (\boldsymbol{\pi} \cdot \boldsymbol{A}^0_\alpha)\right)\boldsymbol{\sigma} + \left(D_\alpha \pi_a + \sqrt{1-\pi^2} A^{a0}_\alpha\right)\boldsymbol{e}_a \tag{2.91}$$

leading to

$$\begin{aligned}\partial_\alpha \boldsymbol{n} \cdot \partial_\alpha \boldsymbol{n} = &\left(\partial_\alpha \sqrt{1-\pi^2} - (\boldsymbol{\pi} \cdot \boldsymbol{A}^0_\alpha)\right)\left(\partial_\alpha \sqrt{1-\pi^2} - (\boldsymbol{\pi} \cdot \boldsymbol{A}^0_\alpha)\right) \\ &+ \left(D_\alpha \pi_a + \sqrt{1-\pi^2} A^{a0}_\alpha\right)\left(D_\alpha \pi_a + \sqrt{1-\pi^2} A^{a0}_\alpha\right)\end{aligned} \tag{2.92}$$

where all repeated indices are summed over. The idea now is to keep *qudratic terms in* $\boldsymbol{\pi}$, this is the proverty of the poor man. This gives

$$\partial_\alpha \boldsymbol{n} \cdot \partial_\alpha \boldsymbol{n} \approx \left(\boldsymbol{\pi} \cdot \boldsymbol{A}^0_\alpha\right)^2 + D_\alpha \pi_a D_\alpha \pi_a + (1-\pi^2) A^{a0}_\alpha A^{a0}_\alpha + \underbrace{2\sqrt{1-\pi^2} A^{a0}_\alpha D_\alpha \pi_a}_{\to 0} \tag{2.93}$$

The last term is dumped because it is the product of a slow and fast piece which is expected to integrate out to zero. Now we realize from eqn. (2.85) that

$$A^{a0}_\alpha A^{a0}_\alpha = \partial_\alpha \boldsymbol{\sigma}(x) \cdot \partial_\alpha \boldsymbol{\sigma}(x) \tag{2.94}$$

This gives us

$$\partial_\alpha \boldsymbol{n} \cdot \partial_\alpha \boldsymbol{n} \approx \partial_\alpha \boldsymbol{\sigma}(x) \cdot \partial_\alpha \boldsymbol{\sigma}(x) + D_\alpha \pi_a D_\alpha \pi_a + A^{a0}_\alpha A^{b0}_\alpha \left(\pi_a \pi_b - \delta_{ab}\pi^2\right) \tag{2.95}$$

Now, interestingly, there is a gauge structure underlying this theory. Recall that the definition of $\boldsymbol{e}_a$ is arbitrary while being "smooth". The quantity

$$\boldsymbol{e}_b \cdot \partial_\alpha \boldsymbol{e}_a = A^{ab}_\alpha \tag{2.96}$$

is a *connection* or *gauge field*. This means suppose we choose a smooth "O(N-1)" field $\mathbf{R}(x)$, the the gauge field will transform as

$$A^{ab}_\alpha(x) = \tilde{A}^{ac}_\alpha(x) R_{cb}(x) + (D_\alpha R^{-1})_{ac} R_{cb}(x) \tag{2.97}$$

leading to a gauge theory for the π fields. We will not pursue this further here, but will come back to this later in a guise called the CP^N model.

Now we assume that $\boldsymbol{\sigma}(x)$ is really small and deviates only minimally from the fully poralized ground state. In this case, A_{ab} are small quantities (kind of proportional to g) hence one can approximate

$$D_\alpha \pi_a = \partial_\alpha \pi_a \tag{2.98}$$

With all of this

$$S[\boldsymbol{n}] \approx \frac{1}{2g}\int \mathrm{d}^d x\, \partial_\alpha \boldsymbol{\sigma}\cdot\partial_\alpha\boldsymbol{\sigma} + \frac{1}{2g}\int \mathrm{d}^d x\, \left[\partial_\alpha \pi_a \partial_\alpha \pi_a + A_\alpha^{a0} A_\alpha^{b0}\left(\pi_a\pi_b - \delta_{ab}\pi^2\right)\right] \tag{2.99}$$

Note that in this form, the action has a very nice structure. The slow fields (A_α^{a0}) appear as sources for the fast fields. By integrating out the fast fields, we can find the interactions between the slow fields induced by the fast fields. Everything is quadratic in the fast fields so this process here is very easy. Taking the first step on "integrate" we use the result over gaussian variables $\boldsymbol{y}$

$$\langle e^{f(\boldsymbol{y})}\rangle = e^{\langle f(\boldsymbol{y})\rangle_{\text{connected}}} \tag{2.100}$$

Now the base action for π-fields is

$$\frac{1}{2g}\int \mathrm{d}^d x\, \partial_\alpha \pi_a \partial_\alpha \pi_a = \frac{1}{2g}\sum_{\tilde{\Lambda}\le|k|\le\Lambda} k^2 \pi_a(k)\pi_a(-k) \tag{2.101}$$

This gives

$$\langle \pi_a(x)\pi_b(x)\rangle = \delta_{ab}\frac{1}{(2\pi)^d}\int_{\tilde{\Lambda}\le|k|\le\Lambda} \mathrm{d}^d k\, \frac{g}{k^2} = \delta_{ab}\, g F_d(\Lambda,\tilde{\Lambda}) \tag{2.102}$$

with

$$F_d(\Lambda,\tilde{\Lambda}) = \frac{S_d}{(2\pi)^d}\int_{\tilde{\Lambda}}^{\Lambda} \mathrm{d}k\, k^{d-3} = \frac{S_d}{(2\pi)^d}\frac{1}{d-2}\left(\Lambda^{(d-2)} - \tilde{\Lambda}^{(d-2)}\right) \tag{2.103}$$

Thus

$$\frac{1}{2g}\int \mathrm{d}^d x\, \left[A_\alpha^{a0} A_\alpha^{b0}\langle\left(\pi_a\pi_b - \delta_{ab}\pi^2\right)\rangle\right] = -\frac{1}{2}(N-2)F_d(\Lambda,\tilde{\Lambda})\int \mathrm{d}^d x\, \partial_\alpha\boldsymbol{\sigma}\cdot\partial_\alpha\boldsymbol{\sigma} \tag{2.104}$$

One gets

$$S_{\text{eff}}[\boldsymbol{\sigma}] = \frac{1}{2}\left(\frac{1}{g} - (N-2)F_d(\Lambda,\tilde{\Lambda})\right)\int \mathrm{d}^d x\, \partial_\alpha\boldsymbol{\sigma}\cdot\partial_\alpha\boldsymbol{\sigma} \tag{2.105}$$

Now we rescale to get

$$S_{\text{new}}[\boldsymbol{\sigma}] = \frac{s^{d-2}}{2}\left(\frac{1}{g} - (N-2)F_d(\Lambda,\tilde{\Lambda})\right)\int \mathrm{d}^d x\, \partial_\alpha\boldsymbol{\sigma}\cdot\partial_\alpha\boldsymbol{\sigma} \tag{2.106}$$

One gets

$$\frac{1}{g(s)} = s^{d-2}\left[\frac{1}{g} - (N-2)\frac{S_d}{(2\pi)^d}\Lambda^{d-2}\left(1 - s^{-(d-2)}\right)\right] \tag{2.107}$$

Writing out a dimensional version using g and defining $K_d = S_d/(2\pi)^d$, we get

$$\frac{1}{\mathrm{g}(s)} = s^{d-2}\left[\frac{1}{\mathrm{g}} - K_d(N-2)(1 - s^{-(d-2)})\right] \tag{2.108}$$

We get the flow equation

$$-\frac{1}{\mathrm{g}(s)^2}\frac{\mathrm{dg}}{\mathrm{d}\ell} = \frac{(d-2)}{\mathrm{g}} - K_d(N-2) \tag{2.109}$$

resulting in

eqn:NLSM:gflow

$$\frac{\mathrm{dg}}{\mathrm{d}\ell} = -(d-2)\mathrm{g} + K_d(N-2)\mathrm{g}^2 \tag{2.110}$$

The ultraviolet flow is described by

$$\mu\frac{\mathrm{dg}}{\mathrm{d}\mu} = (d-2)\mathrm{g} - K_d(N-2)\mathrm{g}^2 \tag{2.111}$$

This last equation can be used to see "asymptotic freedom". Stay in $d = 2$ and start at some scale μ_0. Solving the flow equation we see that

$$\frac{1}{\mathrm{g}(\mu)} - \frac{1}{\mathrm{g}(\mu_0)} = K_d(N-2)\ln\left(\frac{\mu}{\mu_0}\right) \tag{2.112}$$

One sees that as $\mu \to \infty$, we see that $\mathrm{g}(\mu)$ goes to zero as

$$\mathrm{g}(\mu) \to \frac{1}{\ln\left(\frac{\mu}{\mu_0}\right)^{K_d(N-2)}} \tag{2.113}$$

The theory becomes "asympotically free".

2.2 Large N

We will now investigate the NLσM in the large N limit. This will illustrate how the large N works (for a second time), and also give a clearer view or the NLσM.

Let us pose a question to motivate this effort. We have seen that the $O(N)$ NLσMhas fixed point in $d > 2$. We have also that the $O(N)$ ϕ^4 theory has a fixed point for $d < 4$. Are these two fixed points "the same"? More precisely, do these two fixed point describe distinct universality classes, or are they belong to the same universality class?

Start with the NLσM action

$$S[\boldsymbol{n}] = \frac{1}{2g}\int \mathrm{d}^d x \ (\partial_\alpha \boldsymbol{n}\cdot\partial_\alpha \boldsymbol{n} - Bn_1(x))\,, \quad \boldsymbol{n}(x)\cdot\boldsymbol{n}(x) = 1 \tag{2.114}$$

with an ultraviolet momentum cutoff Λ, where B is an external magnetic field. Let us implement the constraint via a Lagrange multiplier (with $\boldsymbol{n}$ now being unrestricted.)

$$\mathcal{Z} = \int D[\lambda]D[\boldsymbol{n}]e^{-S[\boldsymbol{n},\lambda]} \tag{2.115}$$

where

$$S[\boldsymbol{n},\lambda] = \frac{1}{2g}\int \mathrm{d}^d x \ [\partial_\alpha \boldsymbol{n}(x)\cdot\partial_\alpha \boldsymbol{n}(x) - Bn_1(x) + \mathbb{i}\lambda(x)\,(\boldsymbol{n}(x)\cdot\boldsymbol{n}(x) - 1)] \tag{2.116}$$

We can now integrate out the $\boldsymbol{n}$ fields to obtain

$$\mathcal{Z} = \int D[\lambda]D[n_1]e^{-S[n_1,\lambda]} \tag{2.117}$$

where

$$S[n_1,\lambda] = \int \mathrm{d}^d x \ \left[\frac{1}{2g}(\partial_\alpha n_1\partial_\alpha n_1 - Bn_1 + \mathbb{i}\lambda n_1^2) - \frac{\mathbb{i}\lambda(x)}{2g} + \frac{N-1}{2}\ln\det[-\boldsymbol{\nabla}^2 + \mathbb{i}\lambda(x)]\right] \tag{2.118}$$

(some constant terms have been dropped). Now we redefine

$$g \to \frac{g}{N} \tag{2.119}$$

to get

$$S[n_1,\lambda] = N\int \mathrm{d}^d x \ \left[\frac{1}{2g}(\partial_\alpha n_1\partial_\alpha n_1 - Bn_1 + \mathbb{i}\lambda n_1^2) - \frac{\mathbb{i}\lambda(x)}{2g} + \frac{1}{2}\ln\det[-\boldsymbol{\nabla}^2 + \mathbb{i}\lambda(x)]\right] \tag{2.120}$$

We see that in the large N limit, the physics is determined by the saddle point values of λ. Let assume a uniform saddle point to get

eqn:NSLM:SPsig

$$-B + 2m^2\sigma = 0 \tag{2.121}$$

eqn:NSLM:SPm

$$\frac{1}{g} - \sigma^2 = \frac{1}{(2\pi)^d}\int \mathrm{d}^d k \frac{1}{k^2 + m^2} \tag{2.122}$$

where we have redefined uniform saddle point function $\mathbb{i}\lambda = m^2$ and $n_1 = \sigma$.

First, for $d = 2$, with $B = 0$, we get

$$\frac{1}{g} = \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2} \tag{2.123}$$

or,

$$m = \Lambda e^{-\frac{2\pi}{g}} \tag{2.124}$$

One finds that the system is gapped for any value of g. This is exactly what we find from eqn. (2.110) which states that for $d = 2$ there is no phase transition as function of g and that the system is in a massive phase (in otherwords, there is RG flow for any finite value of g.)Find out what happens in $d = 1$.

Now for $d > 2$, with $B = 0$. The "gap equation" becomes (this is approximately true)

$$\frac{1}{g} = \frac{K_d}{d-2}\Lambda^{d-2} - m^{d-2} K_d \frac{\pi}{2 \sin \frac{\pi(d-2)}{2}} + \frac{m^2 K_d}{d-4}\Lambda^{d-4} \tag{2.125}$$

where we have used eqn. (1.96) and eqn. (1.50) One immediately sees that there is a critical point for $d > 2$, this is where the mass vanishes. One get

$$\frac{1}{g_c} = \frac{K_d}{d-2}\Lambda^{d-2} \tag{2.126}$$

precisely as obtained from the RG analysis. Further, we have, in $2 < d < 4$,

$$\frac{1}{g_c} - \frac{1}{g} = \frac{\pi K_d}{2\sin(\pi(d-2)/2)} m^{d-2} \tag{2.127}$$

Defining $t = (g - g_c)/g_c$, we get

$$m \sim t^{1/(d-2)} \Longrightarrow \xi \sim t^{-1/(d-2)} \tag{2.128}$$

We obtain

$$\nu = \frac{1}{d-2} \tag{2.129}$$

precisely as found from the RG analysis. Suddenly, we realize that this is also exactly what we found in the large N analysis of the $O(N)$ ϕ^4 theory! Suggesting that the critical point is in the same universality as the ϕ^4 $O(N)$ theory!

Now, the first saddle point equation eqn. (2.121) with $B = 0$ gives that $\sigma = 0$, when $m \neq 0$. This is a statement that the gapped phase for $g > g_c$ is $O(N)$ symmetric. Interestingly, we see that for $g < g_c$, we get that $m = 0$ and $\sigma \neq 0$, and eqn. (2.122) give

$$\sigma^2 = \frac{1}{g} - \frac{1}{g_c} \Longrightarrow \sigma \sim |t|^{1/2} \tag{2.130}$$

or $\beta = 1/2$! This again, agrees with the full RG result of the previous section. More strikingly, this also agrees with the result for β from the large N ϕ^4 theory.

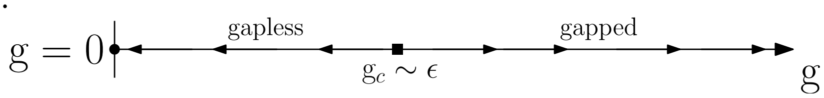
Now that two of the critical exponents are the same for the large N limit of the $O(N)$ NLσM and ϕ^4 theories, it follows from scaling laws that all of the critical exponents are equal. The critical point of both theories belong to the same universality class (different from the Landau meanfield universality class) atleast at sufficiently large N in $d > 2$.

2.3 Upshot

The main results of this chapter are the following. Spatial dimensions $d \leq 2$ are special, in that one cannot have a true long range order at finite temperatures for a system with a continuous symmetry like $O(N), N \geq 2$.

2.3.1 d>2

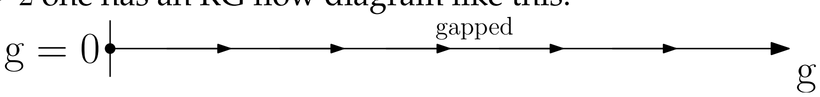
There are two phases for $d > 2$ (and $N > 3$) separated by a critical point g_c which is order $\epsilon = d - 2$ (at one loop level). The phase for $\mathrm{g} > \mathrm{g}_c$ is gapped (finite correlation length), and that for $\mathrm{g} < \mathrm{g}_c$ is a gapless phase (Goldstone modes).



Further, the critical point is the same universality class as the $O(N)$ ϕ^4 theory (atleast at large enough N).

2.3.2 d=2

For $N > 2$ one has an RG flow diagram like this:



In other words, our analysis shows that while Mermin-Wagner theorem does not allow for long range order at finite temperatures, the phase that is realized is gapped, i. e., with a finite correlation length.

For $N = 2$ our RG analysis, at least to one loop level, suggests that there is a scale variant phase, as it predicts now flow for g. What is really going on for $N = 2$ and $d = 2$? This is what we will discuss in the next chapter.

3

Berezinskii-Kosterlitz-Thouless (BKT) Physics

3.1 Whats the hoopla about?

Landau's ideas have influenced much of the thought and understanding of complex systems. One of his key ideas is that symmetry is a central concept that helps describe a phase. The clichéd example is that of a short ranged ferromagnet, a states that breaks the rotational symmetry of the Hamiltonian. Symmetries may be continuous or discrete. In a Heisenberg magnet, the rotational symmetry is continuous and is encoded in the group $SU(2)$. If on the other hand the material has an easy axis, then the spins like to align along this axis, and the $SU(2)$ (or $SO(3)$) symmetry group is then reduced to $\mathbb{Z}_2$, i. e., a discrete symmetry. One could also have an easy plane, where the spins are confined to a plane resulting in a $U(1)$ or $O(2)$ symmetry. $U(1)$ or $O(2)$ is the "simplest" possible continuous symmetry. An array of Josephson junctions is a canonical example of a system with $U(1)$ (or $O(2)$) symmetry.

The well known Glodstone theorem (which we saw in the last chapter), says that if a continuous symmetry is broken, such a symmetry broken ground state has gapless excitations above it. This arises from the "long wavelength functuations" of the "order parameter" field. The intriguing story is that in $d = 2$ (two dimensions), these Goldstone modes from the infrared can wreck havoc. In particular any system with a continuous symmetry cannot have a long range ordered phase at finite temperature – a result usually called the Mermin-Wagner theorem. If one is casual, one may conclude that there are no finite temperature phase transitions in $d = 2$ if there is a continuous symmetry, and this was found less casually via the NLσMapproach for $N > 2$. In other words, this result (no phase transition) is almost true! Well, *almost*! If the symmetry group is $U(1)$, then a phase transition is indeed possible *without* any conflict with the Mermin-Wagner theorem! In other words the transition, the Berezinskii-Kosterlitz-Thouless (BKT) transition, will be between phases that *do not* have any long range order. In other words, BKT transition provides a paradigmatic example of a phase transition without symmetry breaking (no long range order) – this is what the hoopla (well, we now realize that this is *not* hoopla at all!) is about. Quite remarkably, this is an outcome of the *topology* of the group $U(1)$. In particular, $\pi_1(U(1)) = \mathbb{Z}$, and this non-null homotopy is the hero of this story.

In fact, the BKT work brought out the key notion of *topological defects*, which are certain types of excitations. This set a new paradigm to understand many things. In fact, Polyakov's famous results on the the confined nature of compact $U(1)$ gauge theory in three dimensions was inspired by this. Bottomline: BKT ideas set a new direction.

Lets dig in.

3.2 The XY model in $d = 2$

Consider a square lattice whose sites are labeled by i, j and a lattice parameter a_0. At each of the sites that is a "planar spin" (phase of superconducting dot) whose configuration is described by the angle θ that it makes with the x-axis of the square lattice, the configuration of the spin at site i is θ_i. The spins interact with their nearest neighbours; simplest is a ferromagnetic interaction, leading to a Hamiltonian

eqn:BKT:XYHam

$$H = J \sum_{i,\delta} (1 - \cos(\theta_{i+\delta} - \theta_i)) \tag{3.1}$$

where $\delta = x$ or y, a form that anticipates future developments. The system, obviously, has a global $U(1)$ or $O(2)$ symmetry given by $\theta_i \mapsto \theta_i + \phi$, and this leaves that Hamiltonian invariant. As is evident, θ_i themselves are the not the interesting objects; it is the spin at site i described by

$$\boldsymbol{s}_i = e^{\mathrm{i}\theta_i} \equiv (\cos\theta_i, \sin\theta_i) \tag{3.2}$$

One is interested in quantities such as $\langle \boldsymbol{s}_i \cdot \boldsymbol{s}_j \rangle$, i. e., the correlations between spins at different sites.

The ground state is easy to find. Make all the spins parallel, i. .e, choose

$$\theta_i = \vartheta, \tag{3.3}$$

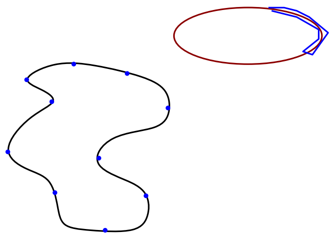


Figure 3.1: Map from $C \mapsto S^1$. Top "spin wave" controur. Bottom: Vortex contour

fig:Contour

without loss of generality we can choose $\vartheta = 0$. What about the excited states? Any state where θ_i differs from its neighbour (but, not by an integral multiple of 2π) will cost energy!

3.2.1 Kinematics

Now, I will attempt to convince you that there are two distinct types of excitations possible. To this end look at the system on scales much larger than the lattice spacing, our model looks like the 2d plane. A lattice point is not described by a 2D vector $\boldsymbol{r}$ and the spin configuration there by $\theta(\boldsymbol{r})$. In this language, the ground state is $\theta(\boldsymbol{r}) = \vartheta$.

Now the excitations have $\theta(\boldsymbol{r})$ which are not homogeneous. Let us consider an arbitrary (non-self intersecting) closed loop C in the plane. Start from an arbitrary point on C and, traverse C say counterclockwise. As we traverse C we obtain a map from $C \mapsto U(1)$. Note that $U(1) = \{e^{\mathbb{i}\theta}\}$ itself can be viewed as the circle S^1. Now, we ask the following question: Does the map from C to S^1 "loop around" S^1? One can answer this question by computing a certain quantity. Let $\boldsymbol{r}_s$ be the starting point on the loop C and $\boldsymbol{r}_e$ be the end point (of course, $\boldsymbol{r}_e = \boldsymbol{r}_s$!). Compute

$$n[C] = \frac{1}{2\pi \mathbb{i}} \int_C \mathrm{d}\boldsymbol{r} \cdot \left(s^*(\boldsymbol{r}) \boldsymbol{\nabla} s(\boldsymbol{r}) \right). \tag{3.4}$$

eqn:WN

One immediately sees that $n[C]$ is *an integer* (no wonder we fell for the notation "n"$[C]$) – how? Note that $s^*(\boldsymbol{r})\boldsymbol{\nabla} s(\boldsymbol{r}) = \mathbb{i}\boldsymbol{\nabla}\theta(\boldsymbol{r})$, so

$$n[C] = \frac{1}{2\pi} \int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla}\theta(\boldsymbol{r}) = \frac{1}{2\pi} \left(\theta(\boldsymbol{r}_e) - \theta(\boldsymbol{r}_s) \right). \tag{3.5}$$

eqn:WNint

It will be hasty to conclude that the right hand side vanishes. Note that what we need is that $s(\boldsymbol{r})$ be a single valued function, which means that

$\theta(\boldsymbol{r}_e)$ *need not be equal* to $\theta(\boldsymbol{r}_n)$, all we need is that $\theta(\boldsymbol{r}_e)$ as obtained from the above equation satisfies

$$\theta(\boldsymbol{r}_e) = \theta(\boldsymbol{r}_s) + 2\pi m, \quad m \in \mathbb{Z} \tag{3.6}$$

Our final result for eqn. (3.7) is

eqn:WN

$$n[C] \in \mathbb{Z}. \tag{3.7}$$

A little reflection will convince you that this integer associated with the contour C tells you how many times you loop around $S^1 \equiv U(1)$ as you go around C.

Suppose, we find that C, as shown fig. 3.1 (top), is of the type that does not loop around S^1, i. e., $n[C] = 0$, we call C a SW contour. Suppose, we find that for the given excitation, *every* contour C is of SW type, we call this excitation (i.e., $\theta(\boldsymbol{r})$) a spin wave excitation.

If a given excitation is not a spin wave excitation, then there is some contour C such that $n[C] \neq 0$. To understand what is going on here, let us start with a premise that everything on the contour C is smooth and nice. What this really means is that $\boldsymbol{\nabla}\theta(\boldsymbol{r})$ that we introduced near eqn. (3.5) is smooth; in fact we even give it its own name

eqn:vgradtheta

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{\nabla}\theta(\boldsymbol{r}) \tag{3.8}$$

Now, we can cast eqn. (3.5) as

$$\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{v} = \int_A \mathrm{d}\boldsymbol{a} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} \tag{3.9}$$

where A is the area enclosed by C, and $\mathrm{d}\boldsymbol{a}$ is the area element. First, lets revisit spin wave excitations from the perspective of $\boldsymbol{v}$. The LHS of the last equation vanishes for every C (by definition), and this implies that

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{0} \quad \text{(spin wave excitation)} \tag{3.10}$$

for a spin wave excitation. For non spin wave excitation, we have

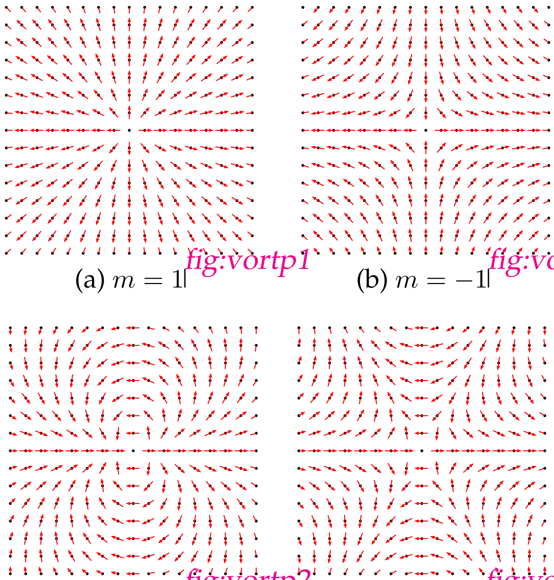
eqn:vortInt

$$\int_A \mathrm{d}\boldsymbol{a} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} = 2\pi m \tag{3.11}$$

where m is a nonzero integer. What does this mean? In other words, what does a non spin wave excitation correspond to?

To understand this, we will first visit the Helmholtz theorem which says that every vector field can be written as

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{\nabla} u + \boldsymbol{\nabla} \times \boldsymbol{w} \tag{3.12}$$



(a) $m = 1$ fig:vortp1 (b) $m = -1$ fig:vortm1

(c) $m = 2$ fig:vortp2 (d) $m = -2$ fig:vortm2

Figure 3.2: Elementary vortex excitations. fig:vort

Since we are in $d = 2$, we see that $\boldsymbol{w} = w(\boldsymbol{r})\boldsymbol{e}_3$ where $\boldsymbol{e}_3$ is a unit vector perpendicular to the plane. In component from we can write this even better:

eqn:Helm

$$v_\alpha = \partial_\alpha u + \varepsilon_{\alpha\beta}\partial_\beta w \tag{3.13}$$

where $\varepsilon_{\alpha\beta}$ is the 2d alternating tensor. In this language eqn. (3.11) can be written as

$$\int_A \mathrm{d}^2 r\, \varepsilon_{\alpha\beta}\partial_\alpha v_\beta = 2\pi m, \tag{3.14}$$

which on using eqn. (3.13) becomes

$$-\int_A \mathrm{d}^2 r \nabla^2 w = 2\pi m. \tag{3.15}$$

These last two equations means that the vorticity or circulation of $\boldsymbol{v}$ enclosed inside C is 2π times an *integer*, i.e., *vorticity is quantized* ! Note that this is true for *all* contours C on which $\boldsymbol{v}(\boldsymbol{r})$ is nice.

We now start wondering what sort of $\boldsymbol{v}(\boldsymbol{r})$ will produce quantized circulation for slightly different contours. Consider the following

eqn:vorty

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{1}{r}\boldsymbol{e}_\phi \tag{3.16}$$

where we use polar coordinate (r, ϕ). We see that *any* contour C that encloses the origin,

$$\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{v} = 2\pi \tag{3.17}$$

Note further that if C does not enclose the origin, then $\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{v} = 0$! Now we can ask, what u and w give the $\boldsymbol{v}$ of eqn. (3.16); it is not difficult to see that

eqn:uw

$$u(r, \phi) = 0, \quad \text{and} \quad w(r, \phi) = \ln(r). \tag{3.18}$$

will do the job. You may have realized the w we found corresponds to the electrostatic potential of a unit charge place at the origin.

How do we understand this from the prespective of eqn. (3.8)? Suppose we try to combine eqn. (3.8) witn eqn. (3.13), then we see that

eqn:thetaw

$$\begin{aligned} \partial_1 \theta &= \partial_2 w \\ \partial_2 \theta &= -\partial_1 w \end{aligned} \tag{3.19}$$

Remembering Cauchy-Reimann equations, one sees that

$$\theta(r, \phi) = \phi \tag{3.20}$$

Thus, $\theta(r, \phi)$ is *multivalued function*, and this corresponds to the elementary vortex state! Do not forget that although $\theta(r, \phi)$ is multivalued, the spin field $e^{\mathrm{i}\theta}$ is single valued.

Fig. 3.2 shows elementary vortex excitations of different quantized vorticities. For an m-vortex, the θ field is given by

$$\theta_{m-\text{vortex}}(r, \phi) = m\phi = m\theta_e(r, \phi) = m\theta_e(\boldsymbol{r}) \tag{3.21}$$

where we have introduced θ_e which is the field of a positive unit vortex.

Some comments are in order.

1. Note that we have "centered" the vortex on a lattice site in fig. 3.2. Consequently, the spin at the origin is "confused" – what this actually means is that the theory has an ultraviolet cutoff and we should not look "too closely" at any spin. We handle this by introducing a vortex core radius a_c. Note that a_c will be of the same order as the lattice constant a (and in fact some authors do not distinguish the two). Also, $\theta(\boldsymbol{r})$ is smooth if one "does not enter" a core region.

2. Any general vortex excitation can now thought of as being made up of many elementary vortices labeled by ℓ. The ℓ-th vortex is located at $\boldsymbol{r}_\ell$ and has a vorticity of m_ℓ, giving us

   eqn:thetav

   $$\theta_v(\boldsymbol{r}) = \sum_\ell m_\ell \theta_e(\boldsymbol{r} - \boldsymbol{r}_\ell). \tag{3.22}$$

Note that, C is a contour that avoids any core region, then

$$\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla}\theta_v(\boldsymbol{r}) = 2\pi \sum_{\ell \text{ inside } C} m_\ell. \tag{3.23}$$

3. If the vortex configuration $V = \{m_\ell, \boldsymbol{r}_\ell\}$ is given, then eqn. (3.22) is one possible realization of this in terms of θ_v. One an add a SW field to this, and this will again correspond to the same configuration of V.

4. What has emerged is that any state $\theta(\boldsymbol{r})$ can be decomposed (not uniquely) as

   eqn:BKT:thetasplit
   $$\theta(\boldsymbol{r}) = \theta_s(\boldsymbol{r}) + \theta_v(\boldsymbol{r}) \tag{3.24}$$

   where $\theta_s(\boldsymbol{r})$ is a *single valued function*, and $\theta_v(\boldsymbol{r})$ is multivalued. With this, we can rewrite eqn. (3.8) as

   eqn:BKT:vsplit
   $$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{\nabla}\theta(\boldsymbol{r}) = \underbrace{\boldsymbol{v}_s(\boldsymbol{r})}_{\boldsymbol{\nabla}\theta_s} + \underbrace{\boldsymbol{v}_v(\boldsymbol{r})}_{\boldsymbol{\nabla}\theta_v} \tag{3.25}$$

   such that $\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{v}_s = 0$ and $\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{v}_v = 2\pi \sum_{\ell \text{ inside } C} m_\ell$.

5. Two excitations $\theta_1(\boldsymbol{r})$ and $\theta_2(\boldsymbol{r})$ are said to be "smoothly deformable to each other" if $\theta_1(\boldsymbol{r}) - \theta_2(\boldsymbol{r})$ is an SW type excitation.

6. An elementary vortex excitation is an infinite two dimensional plane is "*topological*". Suppose, we have a $\theta(\boldsymbol{r})$ such that $\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla}\theta = 2\pi$ if C encloses the origin, i.e., an elementary vortex a the origin. Suppose find a new excitation by a smooth deformation of this excitation, by addition of a spin wave excitation, we see that the vortex at the origin cannot be moved away. In other words, smooth deformations can not get rid of vortices, and hence if a vortex is handed over to us, then no smooth deformation will get rid of it, and is topological in this sense.

   By the same token, one cannot have a single elementary vortex excitation in a 2d box with periodic boundary conditions, again for topological reasons! To see this note that in a periodic box, $\theta(\boldsymbol{r})$ need not be periodic, only $\boldsymbol{s}(\boldsymbol{r})$ has to be periodic. The periodic box is same as a 2-torus. We can always find a contour C that does not enclose any vortices, and thus $\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla}\theta = 0$. Viewing the contour C as the boundary of the complementary region (on the torus), we see that $\int_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla}\theta = 2\pi \sum_\ell m_\ell$, which now includes all the vortices of the excitation. If we assume that there is only one elementary excitation, then we will get m for that excitation is zero, since $0 = 2\pi \sum_\ell m_\ell$. What this means, of course, is that on a torus, the sum total of vorticities of all vortices much vanish.

We now realize that the term "*topological defect*" for a vortex is indeed an apt one! Now a natural question arises. Can vortex excitations be accessed thermally? To answer this question, we need to calculate the energetics of the these types of excitations. It is an appropriate place to make an important comment. We saw in the previous discussion that excitations are spin wave type, or vortex type. That discussion is entirely *kinematical*, in that the possibility of such excitations was deduced solely from the fact that that spin $\boldsymbol{s}(\boldsymbol{r})$ is single valued while $\theta(\boldsymbol{r})$ is multivalued. Now, turning to the energetics, we attempt to understand if the vortex excitations are thermally possible and if so what effect(s) they have on physics of the XY system.

3.2.2 Energetics

The Hamitonian eqn. (3.1) is a pain to handle, in particular the cosine term which respects the fact that θ is really a compact variable. We can take a pragmatic approach to handle this, which entails two physically motivated ideas. (i) We will include vortex excitations (which actually arise from the compact nature of θ) (ii) We will "linearlize" the energetics, in that energy will be proportional to the square of the deformation. Putting these two ideas together, we rewrite eqn. (3.1) as

$$H = \frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\theta = \frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{v} \cdot \boldsymbol{v} \tag{3.26}$$

where $\rho_0 = J$ (in d dimensions $\rho_0 = Ja^{2-d}$, recall previous chapter) is the stiffness. The key point is that θ and $\boldsymbol{v}$ are short forms for the decomposition into spin wave (smooth) and vortex excitations as in eqn. (**??**) and eqn. (**??**).

Okay, lets get to work. We will begin with some manipulations

$$\begin{aligned} H &= \frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, (\boldsymbol{\nabla}\theta_s + \boldsymbol{\nabla}\theta_v)^2 \\ &= \frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, \left[(\boldsymbol{\nabla}\theta_s)^2 + (\boldsymbol{\nabla}\theta_v)^2 + 2\boldsymbol{\nabla}\theta_s \cdot \boldsymbol{\nabla}\theta_v\right] \\ &= H_s + H_v + \rho_0 \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla}\theta_s \cdot \boldsymbol{\nabla}\theta_v \end{aligned} \tag{3.27}$$

Let us analyze the last term. Note that for any vector $\mathbf{c}$,

$$\mathbf{c} \cdot \boldsymbol{\nabla}\theta_v = \mathbf{c} \times \boldsymbol{\nabla} w \tag{3.28}$$

where w is related to θ_v via eqn. (3.19). Thus

$$\begin{aligned}\int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla}\theta_s \cdot \boldsymbol{\nabla}\theta_v &= \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla}\theta_v \times \boldsymbol{\nabla} w \\ &= -\frac{1}{(2\pi)^4} \int \mathrm{d}^2\boldsymbol{q}_1 \mathrm{d}^2\boldsymbol{q}_2\, (\boldsymbol{q}_1 \times \boldsymbol{q}_2)\theta_v(\boldsymbol{q}) w(\boldsymbol{q}) \int \mathrm{d}^2\boldsymbol{r}\, e^{\mathbb{i}(\boldsymbol{q}_1+\boldsymbol{q}_2)\cdot\boldsymbol{r}} \\ &= \frac{1}{(2\pi)^2} \int \mathrm{d}^2\boldsymbol{q}\, (\boldsymbol{q} \times \boldsymbol{q})\theta_v(\boldsymbol{q}) w(-\boldsymbol{q}) \\ &= 0\end{aligned} \tag{3.29}$$

and we get this absolutely nice result

$$H = \underbrace{\frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla}\theta_s \cdot \boldsymbol{\nabla}\theta_s}_{H_s} + \underbrace{\frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla}\theta_v \cdot \boldsymbol{\nabla}\theta_v}_{H_v} \tag{3.30}$$

eqn:BKT:HsHv

that the spin wave "does not talk" to the vortex. Of course, this is a result of "linearizing" the cosine potential. The term H_s is very easy to handle, this is nicely quadratic in θ_s and we can use very standard methods (as we will do below). H_v requires a bit more work. First note that

$$\boldsymbol{\nabla}\theta_v \cdot \boldsymbol{\nabla}\theta_v = \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} w \tag{3.31}$$

as is evident from eqn. (3.19). Now we wish to write an analogous equation as eqn. (3.22) for w. Fist note from eqn. (3.18) that

$$w_e(\boldsymbol{r}) = \ln\left(\frac{|\boldsymbol{r}|}{R}\right) \tag{3.32}$$

and

$$w_{m-\mathrm{vortex}}(\boldsymbol{r}) = m \ln\left(\frac{|\boldsymbol{r}|}{R}\right) \tag{3.33}$$

The analog of eqn. (3.19) is then

$$w(\boldsymbol{r}) = \sum_l m_l \ln\left(\frac{|\boldsymbol{r} - \boldsymbol{r}_\ell|}{R}\right) \tag{3.34}$$

eqn:BKT:wv

Note now an extremely nice result

$$\boldsymbol{\nabla}^2 w = 2\pi \sum_\ell m_\ell \delta(\boldsymbol{r} - \boldsymbol{r}_\ell) \tag{3.35}$$

eqn:Green

We thus see that

$$\begin{aligned} H_v &= \frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} w \\ &= \frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r} \left(\left[\sum_\ell m_\ell^2 (\boldsymbol{\nabla} \ln |\boldsymbol{r} - \boldsymbol{r}_\ell|)^2 \right] + \left[\sum_{\ell' \neq \ell} m_\ell m_{\ell'} \boldsymbol{\nabla} \ln |\boldsymbol{r} - \boldsymbol{r}_\ell| \cdot \boldsymbol{\nabla} \ln |\boldsymbol{r} - \boldsymbol{r}_{\ell'}| \right] \right) \end{aligned} \tag{3.36}$$

eqn:Hvtmp

Evaluation of the above requires two results. First,

$$\int \mathrm{d}^2\boldsymbol{r}\, (\boldsymbol{\nabla} \ln |\boldsymbol{r}|)^2 = 2\pi \int_0^\infty \mathrm{d}r r \frac{1}{r^2} \tag{3.37}$$

We now introduce R, the system size and regulate the short distance by the core radius a_c to get

$$\frac{\rho_0}{2} \int \mathrm{d}^2\boldsymbol{r}\, (\boldsymbol{\nabla} \ln |\boldsymbol{r}|)^2 = \pi\rho_0 \ln\left(\frac{R}{a_c}\right) + E_c, \tag{3.38}$$

eqn:BKT:res1

where E_c is the core energy which subsumes all the ultraviolet physics. The second useful result is

$$\begin{aligned} \int \mathrm{d}^2\boldsymbol{r}\, \boldsymbol{\nabla} \ln |\boldsymbol{r} - \boldsymbol{r}_1| \cdot \boldsymbol{\nabla} \ln |\boldsymbol{r} - \boldsymbol{r}_2| &= \int \mathrm{d}^2\boldsymbol{r}\, \left[\boldsymbol{\nabla} \cdot (\ln |\boldsymbol{r} - \boldsymbol{r}_1| \cdot \boldsymbol{\nabla} \ln |\boldsymbol{r} - \boldsymbol{r}_2|) - \ln |\boldsymbol{r} - \boldsymbol{r}_1| \boldsymbol{\nabla}^2 \ln |\boldsymbol{r} - \boldsymbol{r}_2| \right] \\ &= \left(\int_{\text{Bound}} \mathrm{d}\phi R \boldsymbol{e}_r \cdot \ln |\boldsymbol{r} - \boldsymbol{r}_1| \boldsymbol{\nabla} \ln |\boldsymbol{r} - \boldsymbol{r}_2| \right) - 2\pi \ln |\boldsymbol{r}_1 - \boldsymbol{r}_2| \\ &= 2\pi \ln R - 2\pi \ln |\boldsymbol{r}_1 - \boldsymbol{r}_2| \\ &= 2\pi \ln\left(\frac{R}{a_c}\right) - 2\pi \ln\left(\frac{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}{a_c}\right) \end{aligned} \tag{3.39}$$

eqn:res2

where eqn. (3.35) is used. Using eqn. (**??**) and eqn. (3.39) in eqn. (3.36), we get

$$H_v = -\pi\rho_0 \sum_{\ell \neq \ell'} m_\ell m_{\ell'} \ln\left(\frac{|\boldsymbol{r}_\ell - \boldsymbol{r}_{\ell'}|}{a_c}\right) + \left(\sum_\ell m_\ell^2\right) E_c + \pi\rho_0 \left(\sum_\ell m_\ell\right)^2 \ln\left(\frac{R}{a_c}\right). \tag{3.40}$$

In the thermodynamic limit, only those vortex excitations with a vanishing total vorticity are possible due to the last term.

3.2.3 High Temperature Phase

What is the physics at when $T \gg J$ of the XY-model eqn. (3.1)? We ask how is a spin at a point $\boldsymbol{r} = r\boldsymbol{e}_x$ (r being measured in units of the lattice spacing) correlated with the one a the origin. Thus we need

$$\langle s(\boldsymbol{r})s(\boldsymbol{0})\rangle = \langle \prod_{i=0}^{r} e^{\mathbb{i}(\theta_{i+x}-\theta_i)}\rangle \tag{3.41}$$

where we have numbered the sites along the x-axis starting from 0 at the origin to r at the point $\boldsymbol{r}$. Now $\theta_{i+x} - \theta_i$ is a "bond variable", and these are what determine the energy.

The simple argument now is that is that when $T \gg J$ each bond "will do its thing" and be uncorrelated with others.

$$\langle \prod_{i=0}^{r} e^{\mathbb{i}(\theta_{i+x}-\theta_i)}\rangle = \prod_{i=0}^{r} \langle e^{\mathbb{i}(\theta_{i+x}-\theta_i)}\rangle \tag{3.42}$$

Now, with $K = J/T$,

$$\begin{aligned}
\langle e^{\mathbb{i}\Delta\theta}\rangle = \langle \cos\Delta\theta\rangle &= \frac{\int_0^{2\pi} \mathrm{d}(\Delta\theta) e^{K\cos(\Delta\theta)} \cos(\Delta\theta)}{\int_0^{2\pi} \mathrm{d}(\Delta\theta) e^{K\cos(\Delta\theta)}} \\
&= \frac{\mathrm{d}}{\mathrm{d}K} \ln\left[\int_0^{2\pi} \mathrm{d}(\Delta\theta) e^{-K\cos(\Delta\theta)}\right] = \frac{\mathrm{d}}{\mathrm{d}K} \ln I_0(K) \\
&\approx \frac{K}{2}
\end{aligned} \tag{3.43}$$

One gets, therefore that

$$\langle s(\boldsymbol{r})s(\boldsymbol{0})\rangle \sim \left(\frac{T}{J}\right)^{-|\boldsymbol{r}|} \Longrightarrow e^{-\frac{|\boldsymbol{r}|}{\xi(T)}} \tag{3.44}$$

where

$$\xi(T) \sim \frac{1}{\ln\left(\frac{T}{J}\right)}. \tag{3.45}$$

is the correlation length (in units of the lattice spacing) and vanishes as $T \to \infty$. In other words, at any $T \gg J$, we have a system with a finite correlation length. Colloquially, the high temperature phase is a "gapped phase" (this will be literally true when we make the quantum-classical connection). By the way, this way of analyzing lattice models is quite standard and goes under the name of "strong coupling" expansions (large T).

3.2.4 Low Temperature Phase

In the very low temperature phase $T \ll J$, vortex excitations are extremely unlikely. One can estimate the correlations assuming only spin wave excitations.

First we note some a useful result. Suppose x is a Gaussian random variable with variance $\sigma^2 = \langle x^2 \rangle$, then

$$\langle e^{\mathrm{i}x} \rangle = C \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{x^2}{2\sigma^2}} e^{\mathrm{i}x} = \left[C \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{(x-\mathrm{i}\sigma^2)^2}{2\sigma^2}} \right] e^{-\frac{\sigma^2}{2}} = e^{-\frac{\langle x^2 \rangle}{2}} \tag{3.46}$$

Thus,

$$\left\langle e^{\mathrm{i}(\theta(\boldsymbol{r})-\theta(\boldsymbol{0}))} \right\rangle = e^{-\frac{1}{2}\langle (\theta(\boldsymbol{r})-\theta(\boldsymbol{0}))^2 \rangle} \tag{3.47}$$

We have to, thus, evaluate, ($V = \pi R^2$ is the 2-volume (area) of the system, J_0 is Bessel function)

$$\begin{aligned}
\langle (\theta(\boldsymbol{r}) - \theta(\boldsymbol{0}))^2 \rangle &= \frac{1}{V} \sum_{\boldsymbol{k}_1, \boldsymbol{k}_2} \underbrace{\langle \theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2) \rangle}_{= \frac{T}{\rho_0 |\boldsymbol{k}_1|^2} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2, \boldsymbol{0}}} \left((e^{\mathrm{i}\boldsymbol{k}_1 \cdot \boldsymbol{r}} - 1)(e^{\mathrm{i}\boldsymbol{k}_2 \cdot \boldsymbol{r}} - 1) \right) \\
&= \frac{T}{\rho_0} \frac{1}{(2\pi)^2} \int \mathrm{d}^2\boldsymbol{k}\, \frac{1}{|\boldsymbol{k}|^2} \left(2(1 - \cos(\boldsymbol{k} \cdot \boldsymbol{r})) \right) \\
&= \frac{T}{\rho_0} \frac{1}{2\pi} \int_{\pi/R}^{\pi/a_c} \frac{\mathrm{d}k}{k} (1 - J_0(kr)) \\
&= \frac{T}{\rho_0} \frac{1}{2\pi} \int_{\pi r/R}^{\pi r/a_c} \frac{\mathrm{d}k}{k} (1 - J_0(k))
\end{aligned} \tag{3.48}$$

The last integral can be evaluated approximately. We expect $r \gg a_c$, while $r \ll R$. There is a nice way to evaluate this integral by using the asymptotic forms of the Bessel functions. But there is an even nice way: look at

$$\begin{aligned}
f(|\boldsymbol{r}|) &= \frac{1}{(2\pi)^2} \int \mathrm{d}^2\boldsymbol{k}\, \frac{1}{|\boldsymbol{k}|^2} \left(2(1 - \cos(\boldsymbol{k} \cdot \boldsymbol{r})) \right) \\
\boldsymbol{\nabla}^2 f(|\boldsymbol{r}|) &= \boldsymbol{\nabla}^2 \frac{1}{(2\pi)^2} \int \mathrm{d}^2\boldsymbol{k}\, \frac{1}{|\boldsymbol{k}|^2} \left((2 - \left[e^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})} + e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})} \right]) \right) = \frac{2}{(2\pi)^2} \int \mathrm{d}^2\boldsymbol{k}\, e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})} = 2\delta(\boldsymbol{r}) \\
\Longrightarrow f(|r|) &= \frac{1}{\pi} \ln\left(\frac{r}{R}\right)
\end{aligned} \tag{3.49}$$

Putting all this together, at low temperatures,

$$\langle s(\boldsymbol{r}) s^*(\boldsymbol{0}) \rangle \sim e^{-\frac{T}{2\pi\rho_0} \ln\left(\frac{|\boldsymbol{r}|}{R}\right)} \sim \left(\frac{R}{|\boldsymbol{r}|} \right)^{\frac{T}{2\pi\rho_0}} \tag{3.50}$$

We thus see that the low temperature phase has *power law* correlations with a dimensionless exponent that depends on the temperature!

These are indeed very interesting results, and if you have not seen this stuff before, very puzzling. We have a system that has a finite correlation length at high temperatures, and a power law phase at low temperatures! Note that the is a regime of power law phases for different low temperatures is also consistent with the one loop RG result, that there is no RG flow i.e., the power law ("critical") phase remains so for small changes in g, i. e., temperature. The question is how does the change from the short ranged correlated system to a power law correlated state occur? Note that the power law correlated phase *does not* have long range order, i. e., does not break the $O(2)$ symmetry.

Enter BKT! Our discussion closely follows the KT paper. Suppose, I am in a large but finite sample of radius R. Eqn. (3.38) gives the energy of a single vortex. For a large system, the free energy change associated with the generation of a single vortex is

$$\Delta F = \pi \rho_0 \ln \left(\frac{R}{a_c} \right) - T \ln \left(\frac{R^2}{a_c^2} \right) \tag{3.51}$$

We see that this process is spontaneous when

eqn:BKT:Tv

$$T > T_v = \frac{\pi \rho_0}{2} \tag{3.52}$$

For $T > T_v$ we expect vortices to proliferate and destroy the power law correlated phase.

Although this picture is indeed useful, what really happens is this. Single votrices are expensive, but the system can make vortex-anti-vortex pairs. A pair will have an energy of interaction given by

$$E_{\text{pair}} = 2\pi \rho_0 \ln \left(\frac{L}{a_c} \right) \tag{3.53}$$

where L is the distance between the pairs. At any given temperature, there will be a population of vortex anti-vortex pairs. This population will "screen" the interactions between the a new pair such that the energy actually looks like the

$$E_{\text{pair}} = \frac{2\pi \rho_0}{\varepsilon(L)} \ln \left(\frac{L}{a_c} \right) \tag{3.54}$$

where $\varepsilon(L)$ is the effective (scale-dependent) dielectric constant that accounts for the screening produced by all vortex pairs of size $< L$. $\varepsilon(L)$ is temperature dependent and diverges at a critical temperature making the unbinding of thermal vortices (making their equilibrium $L \to \infty$). When this unbinding takes place, the power law correlated phase gives way to a short range correlated phase. The remarkable thing is that this unbinding takes place at a specific temperature called $T_{\rm BKT}$. How does one implement this physical idea? This was figured out in detail by Kosterliz via RG. He looked at the RG flow of $\rho(L) \equiv \frac{\rho_0}{\varepsilon(L)}$. We shall see this in some depth.

The careful reader may have noticed a crucial difference between this system and the previous ones looked at in this book. We had applied a magnetic field that deliberately breaks the symmetry. We do not attempt to do that here since both sides of the putative transition at a T_v are $U(1)$ symmetric. We need to look for an effective stiffness (or "susceptibility") that respects the internal symmetry. At a microscopic level this is done in the following way. Suppose we go to each bond and say that the phase difference on that bond prefers to have a value $A_{i\delta}$, we write

eqn:BKT:twistedXY

$$H[A] = J \sum_{i\delta} (1 - \cos(\theta_{i+\delta} - \theta_i - A_{i\delta})) \tag{3.55}$$

First note that application of $A_{i\delta}$ still preserves the $U(1)$ global symmetry of the system. If we let $A_{i\delta} = A_\delta$, i. e., a uniform A-field, then we get that ground state has

eqn:BKT:twistedGS

$$\Delta_\delta \theta_i = \theta_{i+\delta} - \theta_i = A_\delta \tag{3.56}$$

and one can see that there is a unique value of θ_i (modulo global $U(1)$ symmetry) in the ground state. (In fact, $A_{i\delta}$ need not be uniform for this to be true; the necessary condition is that $A_{i\delta}$ should not enclose a magnetic flux in any plaquette.). One sees that one can derive a "current" operator via

eqn:BKT:vel

$$j_{i\alpha} = -\left.\frac{\partial H}{\partial A_{i\alpha}}\right|_{A=0} = J \sin(\theta_{i+\alpha} - \theta_i) \tag{3.57}$$

For the ground state described in eqn. (3.56), we see that

$$j_{i\alpha} = J \sin A_\alpha = J A_\alpha \tag{3.58}$$

If we view $A_{i\alpha}$ as a "force", and $j_{i\alpha}$ as "response", then we see that

$$\frac{\partial \langle j_{i\alpha} \rangle}{\partial A_\beta} = \kappa_{\alpha\beta} \underbrace{=}_{\text{for eqn. (3.56)}} J \delta_{\alpha\beta} \tag{3.59}$$

$\kappa_{\alpha\beta}$ is the ("superfluid") stiffness. More generally, we write

$$\begin{aligned} F &= -T \ln \mathcal{Z}[A], \quad \mathcal{Z}[A] = \text{tr}_{\{\theta\}} e^{-H[A]/T} \\ &\approx -V \frac{1}{2} A_\alpha \kappa_{\alpha\beta} A_\beta \\ \implies \kappa_{\alpha\beta} &= \frac{1}{V} \frac{\partial^2 F}{\partial A_\alpha \partial A_\beta} \end{aligned} \qquad \text{eqn:BKT:kappa} \quad (3.60)$$

which is hardly surprising (we did not have a linear term in A as free energy should not depend on the direction of A). Our next task is to obtain an explicit expression for $\kappa_{\alpha\beta}$.

To see what to do in the continuum, we start with the following idea. Suppose, $A_{i\alpha}$ were "small". Then,

$$\begin{aligned} \cos(\Delta_\alpha \theta_i - A_{i\alpha}) &= \cos(\Delta_\alpha \theta_i) \cos(A_{i\alpha}) + \sin(\Delta_\alpha \theta_i) \sin(A_{i\alpha}) \\ &\approx \cos(\Delta_\alpha \theta_i) + \sin(\Delta_\alpha \theta_i) A_{i\alpha} - \frac{1}{2} A_{i\alpha}^2. \end{aligned} \quad (3.61)$$

With this, we get

$$H[A] = J \sum_{i\delta} (1 - \cos(\theta_{i+\delta} - \theta_i)) - \sum_{i\delta} j_{i\delta} A_{i\delta} + \frac{J}{2} \sum_{i\delta} A_{i\delta}^2 \quad (3.62)$$

where the current $v_{i\delta}$ is defined in eqn. (3.57). This, naturally motivates the continuum version, using eqn. (3.30)

$$H[A] = H_s + H_v - \int d^2\boldsymbol{r}\, \boldsymbol{j}(r) \cdot \boldsymbol{A}(\boldsymbol{r}) + \frac{\rho_0}{2} \int d^2\boldsymbol{r}\, \boldsymbol{A}^2(\boldsymbol{r}) \quad (3.63)$$

where we have used the current $\boldsymbol{j} = \rho_0 \boldsymbol{\nabla}\theta$ as simply $\boldsymbol{\nabla}\theta$. We are now ready to obtain $\kappa_{\alpha\beta}$ defined in eqn. (3.66). One finds an explicit expression

$$\begin{aligned} Z[A] &= \int d\{\theta\} e^{-H/T} \left(1 - \frac{\rho_0}{T} \int d^2\boldsymbol{r} \boldsymbol{\nabla}\theta \cdot \boldsymbol{A}(\boldsymbol{r})) + \frac{\rho_0^2}{T^2} \frac{1}{2} \int d^2\boldsymbol{r} d^2\boldsymbol{r}' \boldsymbol{\nabla}_{\boldsymbol{r}}\theta \cdot \boldsymbol{A}(\boldsymbol{r}) \boldsymbol{\nabla}_{\boldsymbol{r}'}\theta \cdot \boldsymbol{A}(\boldsymbol{r}') + \dots \right) \\ &\quad \times e^{\frac{\rho_0}{2} \int d^2\boldsymbol{r}\, \boldsymbol{A}^2(\boldsymbol{r})} \\ &= Z_0 \left[1 - \frac{1}{T} \int d^2\boldsymbol{r} \langle \boldsymbol{\nabla}\theta \rangle \cdot \boldsymbol{A}(\boldsymbol{r}) + \frac{1}{T^2} \int d^2\boldsymbol{r} d^2\boldsymbol{r}' \boldsymbol{A}(\boldsymbol{r}) \cdot \langle \boldsymbol{\nabla}_{\boldsymbol{r}}\theta \otimes \boldsymbol{\nabla}_{\boldsymbol{r}'}\theta \rangle \cdot \boldsymbol{A}(\boldsymbol{r}') \right] \times e^{\frac{\rho_0}{2} \int d^2\boldsymbol{r}\, \boldsymbol{A}^2(\boldsymbol{r})} \end{aligned} \quad (3.64)$$

Since $\langle \boldsymbol{\nabla}\theta \rangle$ vanishes, we have

$$F = F_0 + \frac{\rho_0}{2} \int d^2\boldsymbol{r}\, \boldsymbol{A}^2(\boldsymbol{r}) - \frac{1}{2} \frac{\rho_0^2}{T} \left[\int d^2\boldsymbol{r} d^2\boldsymbol{r}' \boldsymbol{A}(\boldsymbol{r}) \cdot \langle \boldsymbol{\nabla}_{\boldsymbol{r}}\theta \otimes \boldsymbol{\nabla}_{\boldsymbol{r}'}\theta \rangle \cdot \boldsymbol{A}(\boldsymbol{r}') \right] \quad (3.65)$$

When $\boldsymbol{A}$ is taken to be uniform, we get

$$\kappa_{\alpha\beta} = \rho_0\delta_{\alpha\beta} - \frac{\rho_0^2}{TV}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'\,\langle\partial_\alpha\theta(\boldsymbol{r})\partial'_\beta\theta(\boldsymbol{r}')\rangle \quad \text{eqn:BKT:kappa} \tag{3.66}$$

which is not surprising – this is the Kubo formula (of course, what else can it be?). Before proceeding with the evaluation, will take a moment to understand what has happed here. Based on eqn. (3.55) and eqn. (3.56), what we are doing is we are imposing a gridient of θ on the system. If you take a moment, you will see that this messes up the boundary conditions on the periodic box! There is a way out of this, it is to imagine that, the fields θ_s on the boundary is fixed at $\boldsymbol{A}\cdot\boldsymbol{r}$. By construction the θ_v part will be automatically periodic. What this means is, for example,

$$\int_0^L \mathrm{d}x\partial_x\theta_s(x,y) = 0 \tag{3.67}$$

where the box runs from 0 to L along the x-direction, which implies

$$\int \mathrm{d}^2\boldsymbol{r}\partial_\alpha\theta_s(\boldsymbol{r}) = 0 \quad \text{eqn:BKT:FixBD} \tag{3.68}$$

To evaluate the result, we use eqn. (3.25) and eqn. (3.34), i. e.,

$$\partial_\alpha\theta \equiv \partial_\alpha\theta_s + \epsilon_{\alpha\beta}\partial_\beta w \tag{3.69}$$

Thus, we need

$$\frac{\rho_0^2}{TV}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'\,\langle\partial_\alpha\theta_s(\boldsymbol{r})\partial'_\beta\theta_s(\boldsymbol{r}')+\epsilon_{\beta\gamma}\partial_\alpha\theta_s(\boldsymbol{r})\partial'_\gamma w(\boldsymbol{r}')+\partial_\alpha\theta_s(\boldsymbol{r})\partial'_\gamma w(\boldsymbol{r}')+\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\partial_\gamma w(\boldsymbol{r})\partial'_\delta w(\boldsymbol{r}')\rangle \tag{3.70}$$

Let us analyze this term by term. First

$$\int \mathrm{d}\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'\langle\partial_\alpha\theta_s(\boldsymbol{r})\partial'_\beta\theta_s(\boldsymbol{r}')\rangle = 0 \tag{3.71}$$

owing to eqn. (3.68). The second and third terms vanish because vortices and spin waves are uncorrelated. Finally, we note that

$$\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta} \tag{3.72}$$

and this the last term is

$$\begin{aligned}&\frac{\rho_0^2}{TV}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'\langle\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\partial_\gamma w(\boldsymbol{r})\partial'_\delta w(\boldsymbol{r}')\rangle \\ &\quad= \frac{\rho_0^2}{TV}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'\langle\delta_{\alpha\beta}\partial_\gamma w(\boldsymbol{r})\partial'_\gamma w(\boldsymbol{r}') - \partial_\alpha w(\boldsymbol{r})\partial'_\beta w(\boldsymbol{r}')\rangle\end{aligned} \tag{3.73}$$

Introduce Fourier transform,

$$w(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} w(\boldsymbol{k}), \quad w(\boldsymbol{k}) = \frac{1}{\sqrt{V}} \int \mathrm{d}^2\boldsymbol{r} e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} w(\boldsymbol{r}) \tag{3.74}$$

$$\frac{\rho_0^2}{TV} \int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}' \langle \delta_{\alpha\beta}\partial_\gamma w(\boldsymbol{r})\partial'_\gamma w(\boldsymbol{r}') - \partial_\alpha w(\boldsymbol{r})\partial'_\beta w(\boldsymbol{r}')\rangle =$$
$$-\frac{1}{V}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\frac{\rho_0^2}{TV}\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}' e^{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}+\boldsymbol{k}_2\cdot\boldsymbol{r}')}\left(\delta_{\alpha\beta}k_{1\gamma}k_{2\gamma} - k_{1\alpha}k_{2\beta}\right)\langle w(\boldsymbol{k}_1)w(\boldsymbol{k}_2)\rangle\right] \tag{3.75}$$

Now, to perform the $\boldsymbol{r}, \boldsymbol{r}'$ integrals, we define

$$\mathbf{R} = \frac{1}{2}(\boldsymbol{r} + \boldsymbol{r}') \quad \boldsymbol{x} = \boldsymbol{r} - \boldsymbol{r}' \tag{3.76}$$

This change has a unit Jacobian, and so,

$$\begin{aligned}
&-\frac{1}{V}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\frac{\rho_0^2}{TV}\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}' e^{\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r}+\boldsymbol{k}_2\cdot\boldsymbol{r}')}\left(\delta_{\alpha\beta}k_{1\gamma}k_{2\gamma} - k_{1\alpha}k_{2\beta}\right)\langle w(\boldsymbol{k}_1)w(\boldsymbol{k}_2)\rangle\right] \\
&\quad = -\frac{1}{V}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\frac{\rho_0^2}{TV}\int \mathrm{d}\mathbf{R}\mathrm{d}\boldsymbol{x} e^{\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\mathbf{R}+\mathrm{i}((\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}/2)}\left(\delta_{\alpha\beta}k_{1\gamma}k_{2\gamma} - k_{1\alpha}k_{2\beta}\right)\langle w(\boldsymbol{k}_1)w(\boldsymbol{k}_2)\rangle\right] \\
&\quad = \sum_{\boldsymbol{k}}\left[\frac{\rho_0^2}{TV}\int \mathrm{d}\boldsymbol{x} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left(\delta_{\alpha\beta}k_\gamma k_\gamma - k_\alpha k_\beta\right)\langle w(\boldsymbol{k})w(-\boldsymbol{k})\rangle\right] \\
&\quad = \delta_{\boldsymbol{k},\boldsymbol{0}}\frac{\rho_0^2}{T}\left(\delta_{\alpha\beta}k_\gamma k_\gamma - k_\alpha k_\beta\right)\langle w(\boldsymbol{k})w(-\boldsymbol{k})\rangle
\end{aligned} \tag{3.77}$$

where the last step is obtained by a simple manipulation[1]. Now from eqn. (3.34), we see that

$$w(\boldsymbol{k}) = -\frac{2\pi}{|\boldsymbol{k}|^2}\mathcal{M}(\boldsymbol{k}) \tag{3.78}$$

$\mathcal{M}(\boldsymbol{k})$ is the fourier transform of the vortex density

$$\mathcal{M}(\boldsymbol{r}) = \sum_\ell m_\ell \delta(\boldsymbol{r} - \boldsymbol{r}_\ell) \tag{3.79}$$

[1]We use the trick: $\sum_{\boldsymbol{k}} \delta_{\boldsymbol{k},\boldsymbol{0}}\hat{k}_\alpha\hat{k}_\beta = \lim_{k_0\to 0}\int_0^{2\pi}\mathrm{d}\phi\int_0^{k_0}\mathrm{d}k k\left(\frac{1}{\pi k_0^2}\hat{k}_\alpha\hat{k}_\beta\right) = \frac{1}{2}\delta_{\alpha\beta}$.

We thus see that the integral contribution is

eqn:BKT:rhoeff

$$\frac{2\pi^2\rho_0^2}{T}\delta_{\alpha\beta}\lim_{\boldsymbol{k}\to\boldsymbol{0}}\frac{\langle|\mathcal{M}(\boldsymbol{k})|^2\rangle}{|\boldsymbol{k}|^2} \tag{3.80}$$

Thus, we get, ρ the effective stiffness (diagonal part of $\kappa_{\alpha\beta}$ at any finite temperature to be

eqn:BKT:rhofin

$$\rho = \rho_0 - \frac{2\pi^2\rho_0^2}{T}\lim_{\boldsymbol{k}\to\boldsymbol{0}}\frac{\langle|\mathcal{M}(\boldsymbol{k})|^2\rangle}{|\boldsymbol{k}|^2} \tag{3.81}$$

This is a key result. Before we analyze this, we will understand this in a more transparent way.

Work in d dimensions. Consider the following problem. It is possible for "charges" of charge m_ℓ to appear if an energy cost $m_\ell^2 E_c$ is paid. Charges interact with each other via a coulomb Kernel $U_d(|\boldsymbol{r}|)$. If ϵ_0 is the bare dielectric constant, then

$$\boldsymbol{\nabla}^2 U_d(|\boldsymbol{r}|) = -\frac{1}{\epsilon_0}\delta(\boldsymbol{r}) \tag{3.82}$$

i. e., $U_d(|\boldsymbol{r}|)$ is the Green's function of the Laplacian. We can define a charge distribution

$$\mathcal{M}(\boldsymbol{r}) = \sum_\ell m_\ell \delta(\boldsymbol{r}-\boldsymbol{r}_\ell) \tag{3.83}$$

The potential at any point is

$$U(\boldsymbol{r}) = \sum_\ell m_\ell U_d(|\boldsymbol{r}|) \tag{3.84}$$

The total energy is given by

eqn:BKT:CoulombGas

$$H[\{m_\ell\}] = \int \mathrm{d}^d\boldsymbol{r}\,\frac{\epsilon_0}{2}|\boldsymbol{\nabla}U(\boldsymbol{r})|^2 + \sum_\ell m_\ell^2 E_c \tag{3.85}$$

This is called the Coulomb gas. Charges may be thermally excited in the Coulomb gas, and this will change the dielectric constant of the medium. Let $\mathcal{M}_{\text{ext}}(r)$ be an external charge distribution. In vacuum (no thermally excited charges), this will produce a potential

$$|\boldsymbol{k}|^2 U_{\text{ext}}(\boldsymbol{k}) = \frac{1}{\epsilon_0}\mathcal{M}_{\text{ext}}(\boldsymbol{k}) \tag{3.86}$$

When we put this in the external charge distribution in the Coulomb gas, the potential that develops will be different. We can write, and expression for this potential as

$$|\boldsymbol{k}|^2 U_{\text{fin}}(\boldsymbol{k}) = \frac{1}{\epsilon_0}(\mathcal{M}_{\text{ind}}(\boldsymbol{k}) + \mathcal{M}_{\text{ext}}(\boldsymbol{k})) \tag{3.87}$$

But the charge induced is determined by the response of the Coulomb gas to external perturbations. Let the charge response be $\chi(\boldsymbol{k})$. Then

$$\mathcal{M}_{\text{ind}}(\boldsymbol{k}) = \chi(\boldsymbol{k}) U_{\text{ext}} = \frac{\chi(\boldsymbol{k})}{\epsilon_0 |\boldsymbol{k}|^2} \mathcal{M}_{\text{ext}}(\boldsymbol{k}) \tag{3.88}$$

Thus, we see that

$$|\boldsymbol{k}|^2 U_{\text{fin}}(\boldsymbol{k}) = \frac{1}{\epsilon_0}\left[1 + \frac{\chi(\boldsymbol{k})}{\epsilon_0 |\boldsymbol{k}|^2}\right] \mathcal{M}_{\text{ext}}(\boldsymbol{k}) \tag{3.89}$$

We thus see that the effective ($\boldsymbol{k}$) dependent dielectric constant due to the induced charges is

$$\frac{1}{\epsilon(\boldsymbol{k})} = \frac{1}{\epsilon_0}\left[1 + \frac{\chi(\boldsymbol{k})}{\epsilon_0 |\boldsymbol{k}|^2}\right] \tag{3.90}$$

Now, recall that

$$\chi(\boldsymbol{k}) = -\frac{1}{T}\langle \mathcal{M}(\boldsymbol{k})\mathcal{M}(-\boldsymbol{k})\rangle \tag{3.91}$$

where the mean is evaluated over the ensemble defined by the equilibrium Coulomb gas eqn. (3.85). This leads to the effective long wavelength dielectric constant

$$\frac{1}{\epsilon_{\text{eff}}} = \frac{1}{\epsilon_0} - \frac{1}{\epsilon_0^2 T} \lim_{\boldsymbol{k}\to\boldsymbol{0}} \frac{\langle \mathcal{M}(\boldsymbol{k})\mathcal{M}(-\boldsymbol{k})\rangle}{|\boldsymbol{k}|^2} \tag{3.92}$$

Compare this with eqn. (3.80)...and it strikes! The XY model in $d = 2$ is nothing but the Coulomb gas!! We will see the Coulomb gas for any d is an important problem. Understanding the Coulomb gas, therefore, is crucial! I am not entirely sure about this, but it seems that KT were the first to analyze the Coulomb gas in $d = 2$,...in 1970s!! Very surprising!

With these insights, let us calculate eqn. (3.81) explicitly

$$\mathcal{M}(\boldsymbol{k}) = \frac{1}{\sqrt{V}} \sum_{\ell} m_\ell e^{-\mathbb{i}\boldsymbol{k}\cdot\boldsymbol{r}_\ell} \tag{3.93}$$

First, some generalities

$$\begin{aligned} \mathcal{M}(\boldsymbol{k}) &= \frac{1}{\sqrt{V}} \int \mathrm{d}^2\boldsymbol{r}\, e^{-\mathbb{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathcal{M}(\boldsymbol{r}) \\ \Longrightarrow \langle \mathcal{M}(\boldsymbol{k})\mathcal{M}(-\boldsymbol{k})\rangle &= \frac{1}{V}\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}' e^{-\mathbb{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\langle \mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}')\rangle \end{aligned} \tag{3.94}$$

Nothing much so far, we are familiar with this. But now consider for "small" $\boldsymbol{k}$,

$$\begin{aligned} &\frac{\langle \mathcal{M}(\boldsymbol{k})\mathcal{M}(-\boldsymbol{k})\rangle}{|\boldsymbol{k}|^2} \\ &= \frac{1}{|\boldsymbol{k}|^2}\langle\left[\frac{1}{V}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}' \left(1 - \mathbb{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}') - \frac{1}{2}(\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}'))^2 + \dots\right)\mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}')\right]\rangle \end{aligned} \tag{3.95}$$

We have to use some physics to evaluate the quantities. Since having a finite total vorticity will cause a large energy, we expect

$$\left[\int \mathrm{d}^2\boldsymbol{r}\,\mathcal{M}(\boldsymbol{r})\right] = 0. \tag{3.96}$$

Next,

$$\frac{1}{V}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'(\boldsymbol{r}-\boldsymbol{r}')\mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}') = \boldsymbol{0} \tag{3.97}$$

because it is "anti-symmetric" (change $\boldsymbol{r}$ to $\boldsymbol{r}'$). Finally, we have

$$\frac{1}{V}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'(\boldsymbol{r}-\boldsymbol{r}')_\alpha(\boldsymbol{r}-\boldsymbol{r}')_\beta\langle \mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}')\rangle = \frac{1}{2}\delta_{\alpha\beta}\frac{1}{V}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'|\boldsymbol{r}-\boldsymbol{r}'|^2\langle \mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}')\rangle \tag{3.98}$$

Finally, we get

$$\lim_{\boldsymbol{k}\to\boldsymbol{0}} \frac{\langle \mathcal{M}(\boldsymbol{k})\mathcal{M}(-\boldsymbol{k})\rangle}{|\boldsymbol{k}|^2} = -\frac{1}{4}\left[\frac{1}{V}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'|\boldsymbol{r}-\boldsymbol{r}'|^2\langle \mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}')\rangle\right] \tag{3.99}$$

resulting in

eqn:BKT:Kfin

$$K = K_0 + \frac{\pi^2 K_0^2}{2}\left[\frac{1}{V}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'|\boldsymbol{r}-\boldsymbol{r}'|^2\langle \mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}')\rangle\right]. \tag{3.100}$$

where we have defined

eqn:BKT:Kdef

$$K_0 = \frac{\rho_0}{T}, \quad K = \frac{\rho}{T}. \tag{3.101}$$

Now,

$$\frac{1}{V}\int \mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{r}'|\boldsymbol{r}-\boldsymbol{r}'|^2\langle\mathcal{M}(\boldsymbol{r})\mathcal{M}(\boldsymbol{r}')\rangle = \frac{1}{V}\langle\left[\sum_{\ell\ell'} m_\ell m'_\ell|\boldsymbol{r}_\ell-\boldsymbol{r}_{\ell'}|^2\right]\rangle \quad (3.102)$$

To evaluate the expectation value, write the partition function of the Coulomb gas (here in d dimensions)

$$\sum_{N_v=0}^{\infty}\sum_{\{m_\ell\}}\int\prod_{\ell=1}^{N_v}\frac{\mathrm{d}^2\boldsymbol{r}_\ell}{a_c^d}\,e^{-H[\{m_\ell,\boldsymbol{r}_\ell\}]} \quad (3.103)$$

where N_v runs over different vortex sectors. We will now assume that E_c is "very large", so that at most we need to worry about two vortex configurations (single vortex configuration is immediately killed). Also the two vortex configurations have only $m_1 = +1$ and $m_2 = -1$ vortex. Thus

$$\begin{aligned}\frac{1}{V}\langle\left[\sum_{\ell\ell'} m_\ell m'_\ell|\boldsymbol{r}_\ell-\boldsymbol{r}_{\ell'}|^2\right]\rangle &\approx \frac{\frac{2}{V}\int\frac{\mathrm{d}^d\boldsymbol{r}_1}{a_c^d}\frac{\mathrm{d}^d\boldsymbol{r}_2}{a_c^d}(1\times -1)|\boldsymbol{r}_1-\boldsymbol{r}_2|^2 e^{-2E_c/T+2\pi K_0\ln\left|\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{a_c}\right|}}{1+\int\frac{\mathrm{d}^d\boldsymbol{r}_1}{a_c^d}\frac{\mathrm{d}^d\boldsymbol{r}_2}{a_c^d}e^{-2E_c/T+2\pi K_0\ln\left|\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{a_c}\right|}+\ldots} \\ &\approx -e^{-2E_c/T}\frac{2}{a_c^{d-2}}\int\frac{\mathrm{d}^d\boldsymbol{r}}{a_c^d}\left(\frac{|\boldsymbol{r}|}{a_c}\right)^{2-2\pi K_0} \\ &= -e^{-2E_c/T}\frac{2S_d}{a_c^{d-2}}\int_{a_c}^{R}\frac{\mathrm{d}r}{a_c}\left(\frac{r}{a_c}\right)^{d+1-2\pi K_0}\end{aligned} \quad (3.104)$$

Using this with $d = 2$ in eqn. (3.100), we get

$$K = K_0 - 2\pi^3 y^2 K_0^2\int_{a_c}^{R}\frac{\mathrm{d}r}{a_c}\left(\frac{r}{a_c}\right)^{3-2\pi K_0} \quad (3.105)$$

where

$$y = e^{-E_c/T} \quad (3.106)$$

is the vortex fugacity. We have seen that K is like the inverse dielectric constant. We look, therefore, at $1/K$ (note $1/K$ has precisely the meaning of g in the the NLσM), which for small fugacity becomes[2]

$$\frac{1}{K} = \frac{1}{K_0} + 2\pi^3 y^2\int_{a_c}^{\infty}\frac{\mathrm{d}r}{a_c}\left(\frac{r}{a_c}\right)^{3-2\pi K_0} \quad \text{eqn:BKT:Kinvfin} \quad (3.107)$$

[2]We have replaced R by ∞ in these discussions.

This is in the perfect state to apply renormalization group. How does one do that? The idea is that we treat $a_c = \Lambda^{-1}$. By changing a_c to sa_c, $s > 1$, we are asking what is the effective K if we account for all vortex pairs whose size is between a_c and sa_c. We interpret,

$$\frac{1}{K(s=1)} = \frac{1}{K_0} + 2\pi^3 y^2(s=1) \int_{a_c}^{\infty} \frac{\mathrm{d}r}{a_c} \left(\frac{r}{a_c}\right)^{3-2\pi K(s=1)} \tag{3.108}$$

where we have replaced K_0 by $K(s=1)$ in the exponent, this is permissible at low fugacity. Now ask what is the meaning of $K(s)$. This is a coarse grained stiffness, i. e., effective stiffness seen by "objects (vortices)" that are larger than sa_c. In other words, it is the stiffness obtained by "integrating" out all vortex pairs of size $< sa_c$. Thus, we write,

eqn:BKT:Ks

$$\frac{1}{K(s)} - 2\pi^3 y^2(s) \int_{(sa_c)}^{\infty} \frac{\mathrm{d}r}{(sa_c)} \left(\frac{r}{(sa_c)}\right)^{3-2\pi K(s)} = \frac{1}{K_0} \tag{3.109}$$

Note that we can view $(sa_s) = \mu^-1$, and the equation can thus be viewed as a relationship between "bare" $(1/K_0)$ and "renormalized" $(1/K(s))$ quantities. Note that we have also accounted for the fact the second parameter in the theory, viz, the vortex fugacity y also flows. An inspection of the last two equations gives,

eqn:BKT:ys

$$y^2(s) s^{-(4-2\pi K(s))} = y^2(1) = e^{-2E_c/T} \tag{3.110}$$

Thus $K(s)$ is the scale dependent coupling constant, and $y(s)$ as the scale dependent fugacity (related to the cost of injecting a vortex pair of scale larger than sa_c.)

We will now write out a Callan-Symanzik like equation to find the RG flow. Demand,

$$\begin{aligned}
&\left[\frac{1}{K(s)} - 2\pi^3 y^2(s) \int_{(sa_c)}^{\infty} \frac{\mathrm{d}r}{(sa_c)} \left(\frac{r}{(sa_c)}\right)^{3-2\pi K(s)}\right] = 0 \\
\implies &\left[s\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{K} - 2\pi^3 y^2(s)\right] \\
-2\pi^3 &\left[2y(s)s\frac{\mathrm{d}y(s)}{\mathrm{d}s} - y^2(s)\left(4 - 2\pi K(s) + \cancelto{0}{2\pi \ln(s) K^2 s\frac{\mathrm{d}K^{-1}}{\mathrm{d}s}}\right)\right] \int_{(sa_c)}^{\infty} \frac{\mathrm{d}r}{(sa_c)} \left(\frac{r}{(sa_c)}\right)^{3-2\pi K(s)} \\
= 0 &
\end{aligned} \tag{3.111}$$

which (the term cancelled to zero is proportional to y^2, and what is dropped overall is $\sim y^4$) using eqn. (3.109) and eqn. (3.110) give us the famous Kosterlitz-Thouless RG equations

$$s\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{K}\right) = 2\pi^3 y^2(s) \quad \text{eqn:BKT:Kflow} \tag{3.112}$$

$$s\frac{\mathrm{d}y(s)}{\mathrm{d}s} = (2 - \pi K(s))y(s) \quad \text{eqn:BKT:yflow} \tag{3.113}$$

The above flow equations have a remarkable feature. There is a "fixed surface" i e., $y = 0$ for which (for any value of K) the there is no flow. Looking at eqn. (3.113) shows that the value of K is crucial in determining the stability of a point along the $y = 0$ fixed surface. When $(2 - \pi K) < 0$, then the point is stable, however, for $(2 - \pi K) > 0$, the point is unstable. Thus,

$$(K, y = 0) = \begin{cases} \text{stable fixed point} & K < K_c \\ \text{unstable fixed point} & K > K_c \end{cases} \tag{3.114}$$

where

$$K_c = \frac{2}{\pi} \tag{3.115}$$

But K corresponds to

$$K = \frac{\rho}{T} \Longrightarrow T_c = \frac{\pi\rho}{2} \tag{3.116}$$

which is tantalizingly similar to eqn. (3.52), except ρ sits here instead of ρ_0! ✍ **Change all ℓ for vortex index to l.** Let us look at the physics near $(K_c, 0)$. Define,

$$x = (K - K_c) \tag{3.117}$$

The flow equations can be recast as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\ell} &= -2\pi^3 K_c^2 y^2 = -8\pi y^2 \\ \frac{\mathrm{d}y}{\mathrm{d}\ell} &= -\pi x y \end{aligned} \quad \text{eqn:BKT:linflow} \tag{3.118}$$

A strong glare at the equations above yields,

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\left[8y^2 - x^2\right] = 0 \Longrightarrow 8y^2 - x^2 = C \quad \text{eqn:BKT:flowcons} \tag{3.119}$$

where C is a constant. This can be used to deduce the flow as shown in fig. 3.3.

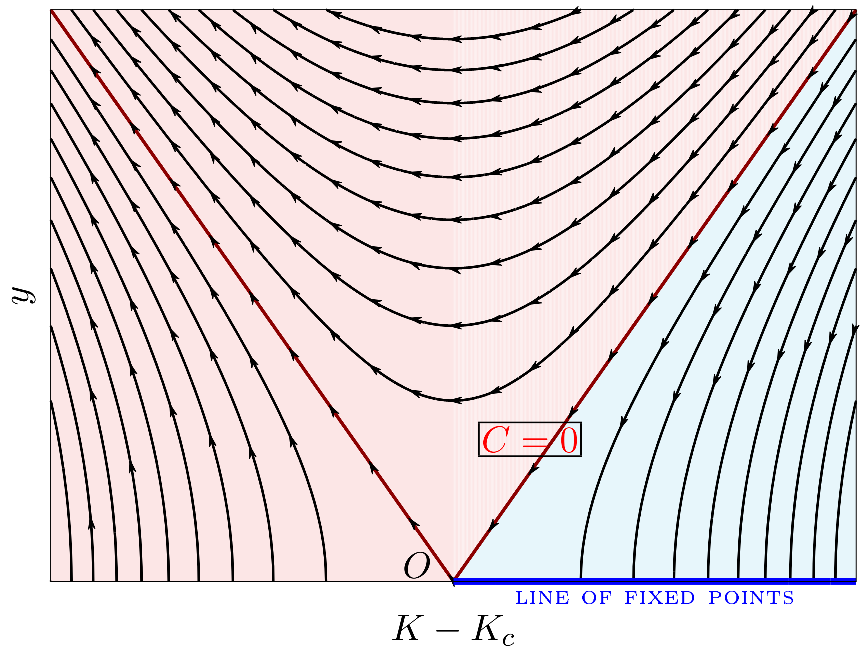


fig:BKT:KtRG
Figure 3.3: Kosterlitz-Thouless renormalization group flow.

We realize that C is a key parameter that determines the physics, and is determined by the initial condition the bare fugacity y_0 and the bare K_0. If (for reasonably close to O), suppose

eqn:BKT:stablebasin
$$C = 8y_0^2 - (K_0 - K_c)^2 < 0, \quad \text{and} \quad K_0 - K_c \geq 0 \tag{3.120}$$

then the RG flow will take this initial point to a point lying on the line of fixed points (see fig. 3.3). Any other initial point (in the light red region) will flow off to a point with $y \to \infty$ and $K \to 0$, i. e., a high temperature fixed point where the fugacity of vortices goes to infinity. The main finding therefore, is that flow takes the system either to the line of fixed points or to the high temperature phase, and the $C = 0$ line (the separatrix) plays a major role. If $C = 0$ with $K_0 - K_c > 0$, the system flows to O. On the other hand for $C = 0$ with $K_0 - K_c < 0$, the system flows to the high temperature state. Thus the point O has one relevant and one irrelevant operator around it! It is natural to view O as the *critical point* that "separates" as fixed point state with powerlaw correlations from the high temperature gapped phase.

It turns out that we can squeeze out a lot more physics from the equations. Given that $C < 0$ flows the system to a the line of fixed points, and $C > 0$ to the gapped high temperature phase, KT ✍ **?Only K** had a beautiful physical idea. They ✍ **?he** argued that C can thought of a parameter like t of the $O(N)$ model (such as defined near eqn. (1.10)). In other words,

$$C = At \tag{3.121}$$

where t is the "distance to the transition" and A is a non-universal constant.

Suppose, we start with a system with bare stiffness ρ_0 which defines the "bare" stiffness $K_0 = \rho_0/T$ and $x_0 = K_0 - K_c$ and bare fugacity y_0. Suppose, also that y_0 is such that

$$C = 8y_0^2 - x_0^2 < 0 \tag{3.122}$$

We see from fig. 3.3 that

$$x(\ell \to \infty) = \sqrt{-C}, \qquad y(\ell \to \infty) = 0 \tag{3.123}$$

Thus

$$K(\ell \to \infty) = K_c + \sqrt{A}\sqrt{|t|} \tag{3.124}$$

The main lesson learnt is that the long wavelength stiffness arriving at the critical point T_{BKT} from the *low temperature side* is $K_c = \frac{2}{\pi}$. We can get further information by studying the flow of x. Using eqn. (3.119) in eqn. (3.118), we get

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\ell} &= -8\pi y^2 = -\pi(C + x^2) \\ \Longrightarrow x(\ell) &= \sqrt{|C|}\left[\frac{(x_0 + \sqrt{|C|}) + (x_0 - \sqrt{|C|})e^{-2\pi\sqrt{|C|}\ell}}{(x_0 + \sqrt{|C|}) - (x_0 - \sqrt{|C|})e^{-2\pi\sqrt{|C|}\ell}}\right] \end{aligned} \tag{3.125}$$

Suppose we start from a regime where x_0 is large (low temperature), i. e., $|C| \ll x_0^2$, this means that the fugacity y_0 is also large (so that $8y_0^2 - x_0^2 = C$), then for "small ℓ" one has $\ell \approx 0$ $x(\ell) \approx x_0$, while for $\ell \to \infty$, we have $x(\ell) \approx \sqrt{|C|}$. At what value of ℓ does the "change take place". We see from the expression of $x(\ell)$ that the condition corresponds to

$$2\pi\sqrt{|C|}\ell_{\text{cross}} \approx 1 \tag{3.126}$$

Thus, the crossover scale factor s_{cross} is

$$s_{\text{cross}} = e^{\ell_{\text{cross}}} = \frac{\xi_{\text{cross}}}{a_c} = e^{\frac{1}{2\pi\sqrt{A}\sqrt{|t|}}} \tag{3.127}$$

It is easily shown that when

$$x(\ell) = \begin{cases} \dfrac{x_0}{1+2\pi x_0 \ell} & , \quad \ell \ll \ell_{\text{cross}} \\ \\ \sqrt{|C|}\left(1+2e^{-2\pi\sqrt{|C|}\ell}\right) & , \quad \ell \gg \ell_{\text{cross}} \end{cases} \tag{3.128}$$

The fugacity goes as

$$y(\ell) = \begin{cases} y_0 - \dfrac{2\pi x_0^3}{y_0}\ell & , \quad \ell \ll \ell_{\text{cross}} \\ \\ \dfrac{\sqrt{|C|}}{2} e^{-\pi\sqrt{|C|}\ell} & , \quad \ell \gg \ell_{\text{cross}} \end{cases} \tag{3.129}$$

The *physics* of the above equations is this. If we view the system on a scale smaller than ξ_{cross}, we will see vortex pair excitations which are governed by x_0 and y_0 (which are taken to be large). However, the fantastic things happen on the longer scales much greater than ξ_{cross}. The effective stiffness renormalizes to $\sqrt{|C|}$ while the vortex fugacity is expoentially small! Vortex pairs have become infinitely expensive. Not only does this provide a clear picture of the physics of the XY model, this also gives a vivid illustration of the beautiful way in which RG works!

The next question is natural. What if $C > 0$ with $x_0 > 0$? Fig. 3.3 tells us that we will flow off to the high temperature gapped phase. Let us investigate. Here

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\ell} &= -\pi(C+x^2) \\ \implies \frac{1}{\sqrt{|C|}}\left[\tan^{-1}\left(\frac{x(\ell)}{\sqrt{C}}\right) - \tan^{-1}\left(\frac{x_0}{\sqrt{C}}\right)\right] &= -\pi\ell \end{aligned} \tag{3.130}$$

Since we have started with $x_0 > 0$, we get that as ℓ increases, $x(\ell)$ eventually has to become negative. Suppose we start with $x_0 \gg \sqrt{C}$, for this, $\tan^{-1}\left(\frac{x_0}{\sqrt{C}}\right) \approx \pi/2$. A characteristic ℓ can be found by demanding that $|x(\ell)|$ also becomes large compared to $\sqrt{C}$, thus giving

$$\frac{1}{\sqrt{C}} = \ell_{\text{corr}} \tag{3.131}$$

This leads naturally to the definition of a correlation length

$$\frac{\xi}{a_c} = e^{\ell_{\text{corr}}} = e^{\frac{1}{\sqrt{A}\sqrt{t}}} \tag{3.132}$$

Treating t as the distance to the transition, we see how the correlation length diverges when the critical point is approached from the high temperature side. While this analysis is very illuminating, the flow takes us to regimes where the linearized approximation eqn. (3.118) breaks down. However, we know qualitatively from fig. 3.3 that when $C > 0$, $K(\ell) \to 0$ as $\ell \to \infty$.

$$\lim_{\ell\to\infty} K(\ell) = \begin{cases} K_c & C = 0^- \text{ i. e., } t = 0^- \\ 0 & C = 0^+ \text{ i. e., } t = 0^+ \end{cases} \tag{3.133}$$

This is a truly *spectacular* result! To see its "spectacularity", suppose in a large thermodynamic system, the critical temperature be T_{BKT} (*NOT* universal). Then what we find is that if you tune the system to $T = T^-_{\text{BKT}}$, then the renormalized infrared stiffness at this point $\rho(T^-_{\text{BKT}})$ is such that

$$\rho(T^-_{\text{BKT}}) = \frac{2}{\pi} T_{\text{BKT}} \tag{3.134}$$

Now, if you tune the system to a temperature $T = T^+_{\text{BKT}}$, then the corresponding long wavelength stiffness is

$$\rho(T^+_{\text{BKT}}) = 0 \tag{3.135}$$

This spectacular prediction made is that the long wavelength stiffness has a *jump discontinuity* across the transition! The jump in the siffness, satisfies a *universal relation*:

$$\frac{\rho(T^-_{\text{BKT}}) - \rho(T^+_{\text{BKT}})}{T_{\text{BKT}}} = \frac{2}{\pi} \tag{3.136}$$

This spectacular prediction has now been verified in many physical systems.