Functional methods I: Lagrangian Quantum Field Theory

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1 Generating functional for correlation functions

Consider a scalar QFT, defined by the action functional,

$$I[\phi(x)] = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right]. \tag{1}$$

The Classical dynamics is described by the equation of motion,

$$\partial^2\phi + \frac{\partial V(\phi)}{\partial\phi} = 0. \tag{2}$$

However all the information of the quantum dynamics are contained in the infinite set of n-point **time ordered** Green's functions/correlation functions,

$$G^{(n)}(x_1, \ldots, x_n) \equiv \langle T\phi(x_1)\ldots\phi(x_n)\rangle,$$

for arbitrary n. Note that there are actually $n!$ terms on the rhs of the above equation, since there are $n!$ number of possible time orderings of n-points.

One can gather together all the Green's functions together into a single "generating functional of Green's functions", $Z[J(x)]$. This is a Taylor series (in powers of $J(x)$[1]), the expansion coefficients of which are the n-point Green's functions of the theory,

$$Z[J(x)] \equiv \langle T\ \exp\left(-i\int d^D x\, J(x)\,\phi(x)\right)\rangle \equiv \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^D x_1 \ldots d^D x_n\, G^{(n)}(x_1, \ldots, x_n)\ J(x_1)\ldots J(x_n),$$

$$G^n(x_1, \ldots x_n) = \left[\frac{\delta}{i\,\delta J(x_1)}\cdots\frac{\delta}{i\,\delta J(x_n)} Z[J]\right]_{J=0}. \tag{3}$$

Thus the aim of solving the QFT (which is to compute n-point time-ordered Green's functions for arbitrary n) can be readily accomplished by solving for the generating functional, $Z[J]$, once and for all. To this end one needs to find the equation obeyed by $Z[J]$, which is known as the Schwinger-Dyson (SD) equation.

[1] For reasons which will become apparent, $J(x)$ is dubbed the "Schwinger source" function. Since its a function, it is not quantized, i.e. it is a classical object.

Exercise:

A. Show that when the source, $J(x)$ is not set to zero, i.e. for $J \neq 0$, the functional derivative of the generating functional is,

$$\frac{\delta}{i\delta J(x)} Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^D x_1 \ldots d^D x_n \, \langle T\phi(x)\phi(x_1)\ldots\phi(x_n)\rangle \, J(x_1)\ldots J(x_n) \tag{4}$$

B. Generalize this to an arbitrary power or polynomial of, $F(\phi(x)) = \sum a_n \, \phi^n(x)$,

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^D x_1 \ldots d^D x_n \, \langle T \, \underline{F\,(\phi(x))}\phi(x_1)\ldots\phi(x_n)\rangle \, J(x_1)\ldots J(x_n) = F\left(\frac{\delta}{i\delta J(x)}\right) Z[J]\,. \tag{5}$$

2 Schwinger-Dyson (SD) Equation for Scalar field theory

To set up the SD equation, we first see how the Green's functions propagate. The first non-trivial case is thus the two point function. For this we first compute the time derivative

$$\begin{aligned}
\partial_0 \langle T\, \phi(x)\phi(y)\rangle &= \partial_0 \Big(\theta(x^0 - y^0)\langle\phi(x)\phi(y)\rangle + \theta(y^0 - x^0)\langle\phi(y)\phi(x)\rangle\Big) \\
&= \langle T\, \dot\phi(x)\phi(y)\rangle + \delta(x^0 - y^0)\,\langle[\phi(x), \phi(y)]\rangle \\
&= \langle T\, \dot\phi(x)\phi(y)\rangle.
\end{aligned}$$

The second term in line 2 containing a delta function vanishes on account of the equal time commutation relation. Then we take a further time derivative

$$\begin{aligned}
\partial_0^2 \langle T\, \phi(x)\phi(y)\rangle &= \partial_0 \langle T\, \dot\phi(x)\phi(y)\rangle \\
&= \langle T\, \ddot\phi(x)\phi(y)\rangle + \delta(x^0 - y^0)\left\langle\left[\dot\phi(x), \phi(y)\right]\right\rangle \\
&= \langle T\, \ddot\phi(x)\phi(y)\rangle - i\,\delta^4(x-y).
\end{aligned} \tag{6}$$

Here again we simplified the second term containing the delta function using the equal time commutation relation,

$$\left[\dot\phi(x), \phi(y)\right]_{x^0 = y^0} = i\,\delta^3(\mathbf{x} - \mathbf{y}).$$

One can then compute easily the Laplacian acting on the two point function to get,

$$\boldsymbol{\nabla}^2 \langle T\, \phi(x)\phi(y)\rangle = \langle T\, \boldsymbol{\nabla}^2\phi(x)\,\phi(y)\rangle. \tag{7}$$

Thus the d'Alembertian operator acting on the time-ordered two point function turns out,

$$\partial^2 \langle T\, \phi(x)\phi(y)\rangle = \langle T\, \partial^2\phi(x)\,\phi(y)\rangle - i\,\delta^4(x-y).$$

Exercise:

A. Show that for time-ordered three point function,

$$\begin{aligned}\langle T\,\phi(x)\,\phi(y)\,\phi(z)\rangle \;\equiv\;& \theta\left(x^0-y^0\right)\theta(y^0-z^0)\langle\phi(x)\,\phi(y)\,\phi(z)\rangle+\theta\left(x^0-z^0\right)\theta(z^0-y^0)\langle\phi(x)\,\phi(z)\,\phi(y)\rangle\\ &+\theta\left(y^0-z^0\right)\theta(z^0-x^0)\langle\phi(y)\,\phi(z)\,\phi(x)\rangle+\theta\left(y^0-x^0\right)\theta(x^0-z^0)\langle\phi(y)\,\phi(x)\,\phi(z)\rangle\\ &+\theta(z^0-x^0)\theta\left(x^0-y^0\right)\langle\phi(z)\,\phi(x)\,\phi(y)\rangle+\theta(z^0-y^0)\theta\left(y^0-x^0\right)\langle\phi(z)\,\phi(y)\,\phi(x)\rangle,\end{aligned}$$

the action of the d'Alembertian is,

$$\partial^2\langle T\,\phi(x)\,\phi(y)\,\phi(z)\rangle=\langle T\,\partial^2\phi(x)\,\phi(y)\,\phi(z)\rangle-i\,\delta^4(x-y)\langle\phi(z)\rangle-i\,\delta^4(x-z)\langle\phi(z)\rangle.$$

B. Generalize the above to a general $(n+1)$-point function using induction

$$\partial^2\langle T\,\phi(x)\,\phi(y_1)\dots\phi(y_n)\rangle=\langle T\,\partial^2\phi(x)\,\phi(y_1)\dots\phi(y_n)\rangle-i\sum_{i=1}^{n}\delta^4(x-y_i)\langle T\phi(y_1)\dots\phi(y_{i-1})\phi(y_{i+1})\dots\phi(y_n)\rangle \tag{8}$$

Applying ∂^2 to both sides of Eq. (4) and then plugging in the result (8) as well as using the equation of motion (2), we get,

$$\begin{aligned}\partial^2\frac{\delta}{i\delta J(x)}Z[J] &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^Dy_1\dots d^Dy_n\,\partial^2\left\langle T\phi(x)\phi(x_1)\dots\phi(x_n)\right\rangle J(y_1)\dots J(y_n)\\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^Dy_1\dots d^Dy_n\,J(y_1)\dots J(y_n)\,\Big[\langle T\,\partial^2\phi(x)\,\phi(y_1)\dots\phi(y_n)\rangle\\ &\qquad\qquad -i\sum_{i=1}^{n}\delta^4(x-y_i)\langle T\phi(y_1)\dots\phi(y_{i-1})\phi(y_{i+1})\dots\phi(y_n)\rangle\Big]\\ &= -\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^Dy_1\dots d^Dy_n\,J(y_1)\dots J(y_n)\,\langle T\,\frac{\partial V(\phi(x))}{\partial\phi(x)}\,\phi(y_1)\dots\phi(y_n)\rangle+J(x)\,Z[J]\\ &= \left[-\frac{\partial V}{\partial\phi}\left(\frac{\delta}{i\delta J(x)}\right)+J(x)\right]Z[J(x)]\,.\end{aligned}$$

Thus we have arrived at the Schwinger-Dyson equation for the generating functional of this QFT,

$$\left[\partial^2\frac{\delta}{i\delta J(x)}+\frac{\partial V}{\partial\phi}\left(\frac{\delta}{i\delta J(x)}\right)-J(x)\right]Z[J]=0 \tag{9}$$

Note that this is a functional differential equation. Also the polynomial in ϕ, $V'(\phi)$ has been replaced by a polynomial in derivatives, $V'(\frac{\delta}{i\delta J})$, thus rendering the differential equation linear.

Note that if we define, an expectation value of in the presence of source, J as,

$$\langle\phi(x)\rangle_J=\frac{\delta Z[J]}{i\delta J(x)},$$

then the above SD equation turns out to be

$$\partial^2 \langle \phi(x) \rangle_J + \left\langle \frac{\partial V}{\partial \phi} \left(\phi(x) \right) \right\rangle_J = J(x),$$

which is just the same equation of motion as the scalar but with a source term in the rhs. Thus J is justified to be dubbed as a source.

3 Solution of the SD-equation : Functional representation of the generating functional (Feynman Path Integral)

As in the case with any linear differential equation, the first step in solving it is to switch to functional Fourier transformed variables,

$$Z\left[J(y)\right] = \int \left[d\varphi(y) \right] e^{i \int d^D y \, J(y) \varphi(y)} \, \tilde{Z}\left[\varphi(y) \right]$$

The measure $[d\varphi(x)]$ is a measure on the space of functions, $\varphi(x)$ as as such is a formal device which is not mathematically well defined (It can only be defined on a lattice i.e. thru a regulator, and then the limit of vanishing lattice spacing). Second thing to note is that $\varphi(x)$ is an integration variable which is not the scalar field, $\phi(x)$, yet. However, we will see it can be identified with the "off-shell"scalar field $\phi(x)$ (courtesy Feynman's insight about the functional integral as representing a sum over histories aka the path integral) and hence we will end up swapping ϕ for φ in the final expression. We get,

$$\int \left[d\varphi \right] \; e^{i \int d^D y \, J(y) \varphi(y)} \left[\partial^2 \varphi + \frac{\partial V}{\partial \phi} \left(\varphi \right) + J(x) \right] \tilde{Z}\left[\varphi \right] = 0 \tag{10}$$

Now recall that derivative of a definite integral is zero,

$$\frac{\delta}{\delta \varphi(x)} Z\left[J \right] = 0,$$

which implies,

$$J(x) \tilde{Z}\left[\varphi \right] = i \frac{\delta}{\delta \varphi(x)} \tilde{Z}\left[\varphi \right],$$

which we plug in Eq. (10) and get the Fourier transformed SD-equation,

$$\left[\partial^2 \varphi + \frac{\partial V}{\partial \phi} \left(\varphi \right) + i \frac{\delta}{\delta \varphi(x)} \right] \tilde{Z}\left[\varphi \right] = 0. \tag{11}$$

This is a first order (functional) differential equation and can be solved using an integrating factor (complete this in the following exercise). The solution is,

$$\tilde{Z}\left[\varphi \right] = \mathcal{N} e^{-i \int d^D x \, \left[\varphi \partial^2 \varphi + V(\varphi) \right]} = e^{i \, I[\varphi]}, \tag{12}$$

where, I is the action functional, (1) and $\mathcal{N}$ is some integration constant.

Exercise:

Complete the derivation (12) from (11).

Thus finally we have the solution to the SD-equation,

$$Z\left[J\right]=\mathcal{N}\int\left[d\varphi\right]\ e^{iI[\varphi]+\int d^Dy\ J(y)\varphi(y)}.$$

The boundary condition, $Z\left[J=0\right]=1$ implies determines the hitherto undetermined integration constant,

$$\mathcal{N}=\frac{1}{\int\left[d\varphi\right]\ e^{iI[\varphi(y)]}}.$$

Now we can swap, φ with our field variables, ϕ as we see that the action for ϕ makes an appearance,

$$Z\left[J(x)\right]=\mathcal{N}\int\left[d\phi\right]\ e^{iI[\phi(x)]+\int d^Dy\ J(x)\phi(x)}. \tag{13}$$

Feynman first obtained such functional expressions as "vacuum to vacuum amplitudes" in the presence of a source, J (creating or destroying ϕ excitations),

$$Z\left[J\right]=\langle 0|0\rangle_J$$

and he interpreted the functional integral as a weighed sum over paths/ sum over histories with the weight of a path/ history being the phase, $\exp\left(i\,I\right)$ i.e. the exponential of the action evaluated on that path. Note that functional integral is over all ϕ i.e. these paths are arbitrary i.e. they do not have to obey the classical equation of motion. Further, noting that taking (powers of) functional derivatives wrt J, adds (powers of) ϕ to the functional integrand,

$$\left(\frac{\delta^n}{i\delta J(x_1)\ldots\delta J(x_n)}\right)Z[J]=\mathcal{N}\int\left[d\varphi\right]\ e^{iI[\varphi]+\int d^Dy\ J(y)\varphi(y)}\ \phi(x_1)\ldots\phi(x_n),$$

and then using Eq.(3), we get a functional integral representation of time-ordered Green's functions,

$$\langle T\phi(x_1)\ldots\phi(x_n)\rangle=\mathcal{N}\int\left[d\varphi\right]\ \phi(x_1)\ldots\phi(x_n)\,e^{iI[\varphi]}. \tag{14}$$

Exercise: Free Field Theory

A. Show that for free theory (denoted by subscript 0), i.e. when

$$V_0(\phi)=\frac{1}{2}m^2\phi^2,$$

the functional integration (13) can be carried out to entirely to solve,

$$Z_0\left[J\right]=\exp\left(-\frac{i}{2}\int d^Dx\,d^Dy\ J(x)\,\Delta_F(x-y)\,J(y)\right) \tag{15}$$

where is $i\Delta_F$ the time-ordered two point function for free fields aka the Feynman propagator aka the causal Green's function.

B. Using (15) show that for free fields

$$\langle T\phi(x_1)\dots\phi(x_n)\rangle = 0, \quad n = \text{odd}.$$

C. If $n = 2m$, i.e. even, then the rhs is a product of propagators,

$$\langle T\phi(x_1)\dots\phi(x_4)\rangle = i\Delta_F(x_1,x_2)\, i\Delta_F(x_3,x_4) + i\Delta_F(x_1,x_3)\, i\Delta_F(x_2,x_4) + i\Delta_F(x_1,x_4)\, i\Delta_F(x_2,x_3).$$

(For general $n = 2m$ there are $\frac{(2m-1)!}{2^{m-1}(m-1)!}$ terms of corresponding to various ways of making m pairs out of total $2m$ objects). Express this in terms of diagrams (these are the position space Feynman diagrams).

4 Interacting Fields and Feynman diagram expansions

For Interacting theories, for which, $V(\phi) = \frac{m^2}{2}\phi^2 + \sum \frac{\lambda_n}{n!}\phi^n$, one can show (by noting that inside the functional integral ϕ can be replaced by)

$$Z[J] = Z[J] = \mathcal{N}\int [d\varphi]\, e^{i\int d^Dx\, \frac{1}{2}\left((\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \sum \frac{\lambda_n}{n!}\phi^n\right) + \int d^Dy\, J(y)\varphi(y)} = \exp\left(-i\sum_n \frac{\lambda_n}{n!}\int d^dx\, \left(\frac{\delta}{i\delta J(x)}\right)^n\right) Z_0[J]\,.$$