Lecture 3

Wave Nature of Light II

ECE 325

OPTOELECTRONICS

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Kasap–1.1B, 1.4, 1.7, 1.9 and 1.10

AR Coatings on Solar Cells

When light is incident on the surface of a semiconductor it becomes partially reflected. The refractive index of Si is about 3.5 at wavelengths around 700 - 800 nm. Thus the reflectance with n_1(air) = 1 and n_2(Si) ≈ 3.5 is

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1-3.5}{1+3.5}\right)^2 = 0.309$$

≈31% of the light is reflected and is not available for conversion to electrical energy; a considerable reduction in the efficiency of the solar cell.

AR Coatings on Solar Cells

We can coat the surface of the semiconductor device with a thin layer of a dielectric material such as Si_3N_4 (silicon nitride) that has an intermediate refractive index.

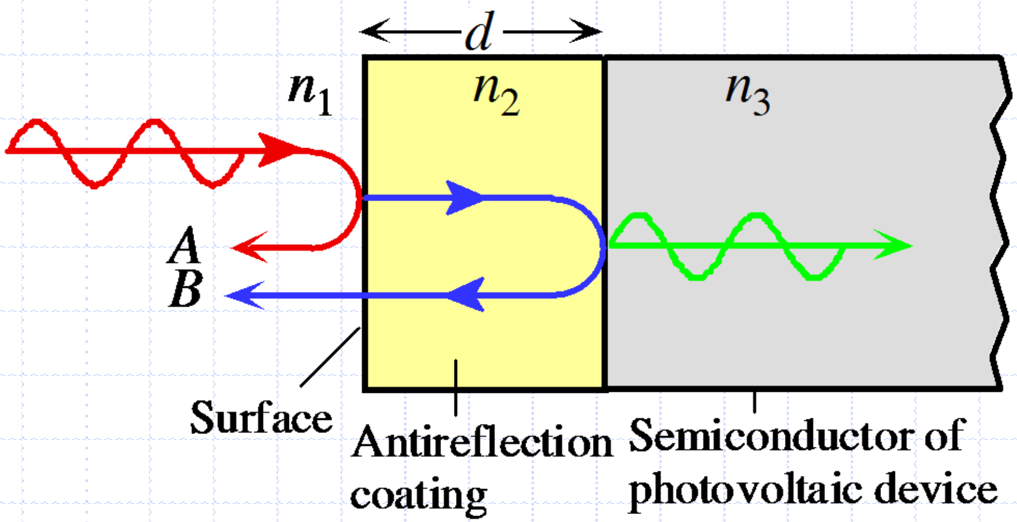


Illustration of how an antireflection coating reduces the reflected light intensity.

In this case
n_1(air) = 1, n_2(coating) ≈ 1.9 and n_3(Si) = 3.5

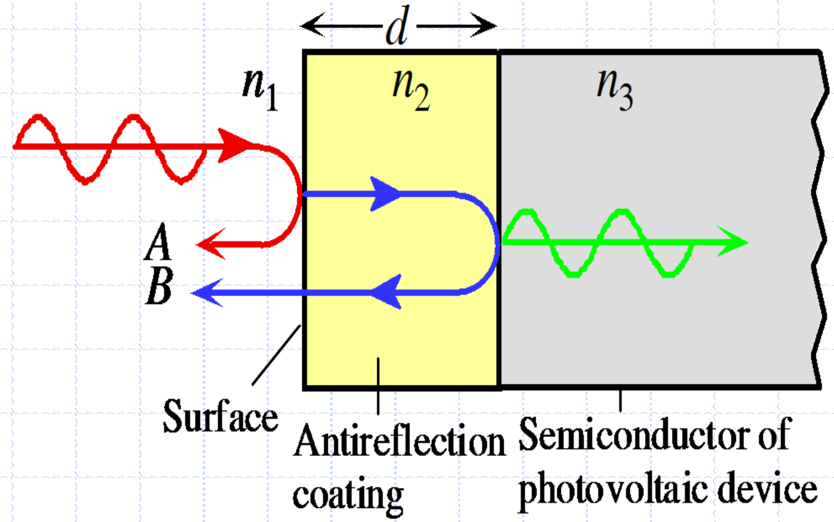
Light is first incident on the air/coating surface. Some of it becomes reflected as *A* in the figure.

Wave *A* has experienced **a 180° phase change** on reflection because this is an **external reflection ($n_2 > n_1$)**.

Wave *B*: reflected at the coating/semiconductor surface with phase change = 180° (**$n_3 > n_2$**)

AR Coatings on Solar Cells

When B reaches A, it has suffered a total delay of traversing the thickness d of the coating twice.



The **phase difference** is equivalent to

$$\Delta\phi = 2\pi \frac{2d}{\lambda_c} = k_c(2d) \quad (k_c = 2\pi/\lambda_c \text{ is the wave number in the coating})$$

$\lambda_c = \lambda / n_2$ (λ_c is the wavelength in the coating)

$$\Rightarrow \Delta\phi = (n_2 k)(2d) = \left(\frac{2\pi n_2}{\lambda}\right)(2d)$$

To reduce the reflected light, A and B **must interfere destructively.** This requires the **phase difference to be π** or **odd-multiples of π, $m\pi$** where **m = 1,3,5**,…is an odd-integer.

AR Coatings on Solar Cells

Destructive interference requires

$$\text{Phase change} = \Delta\phi = (n_2 k)(2d) = m(\pi)$$

***m* = 1,3,5…odd integer**

$$\left(\frac{2\pi n_2}{\lambda}\right)2d = m\pi \quad \text{or} \quad d = m\left(\frac{\lambda}{4n_2}\right)$$

Thus, the thickness of the coating must be **odd multiples** of the quarter wavelength in the coating and depends on the wavelength

To obtain a good degree of destructive interference between waves *A* and *B*, the two amplitudes must be **comparable**.

We need $\boldsymbol{n_2 = \sqrt{n_1 n_3}}$

AR Coatings on Solar Cells

For a Si solar cell, √(3.5) or 1.87. Thus, Si_3N_4 is a good choice as an **antireflection coating material** on Si solar cells.

Taking the wavelength to be 700 nm,

d = (700 nm)/[4 (1.9)] = **92.1 nm**

or **odd-multiples** of d.

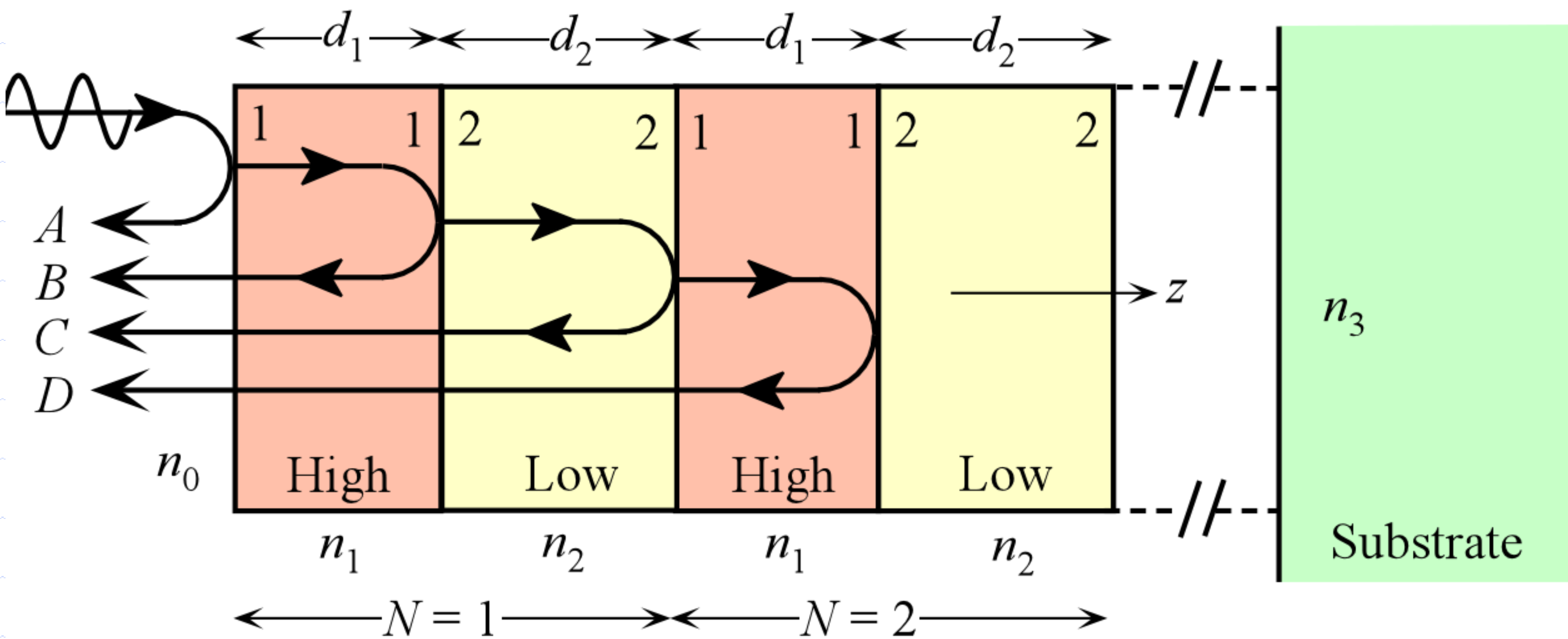
$$R_{\min} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3} \right)^2$$

$$R_{\min} = \left(\frac{1.9^2 - (1)(3.5)}{1.9^2 + (1)(3.5)} \right)^2 = 0.00024 \; or \; 0.24\%$$

Reflection is almost entirely extinguished

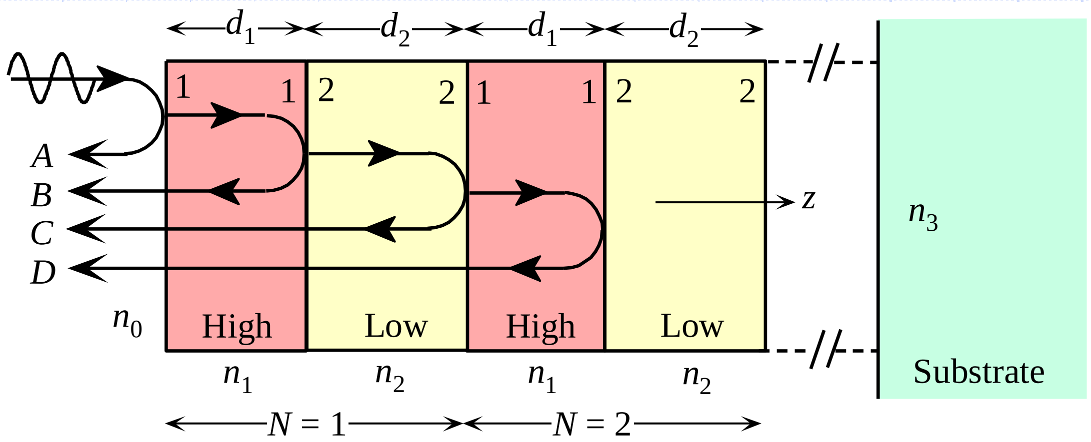
However, only at 700 nm.

Dielectric Mirror or Bragg Reflector



Schematic illustration of the principle of the dielectric mirror with many low and high refractive index layers

Dielectric Mirror or Bragg Reflector



A **dielectric mirror** has a stack of dielectric layers of alternating refractive indices.

Let $n_1(= n_H) > n_2(N_L)$

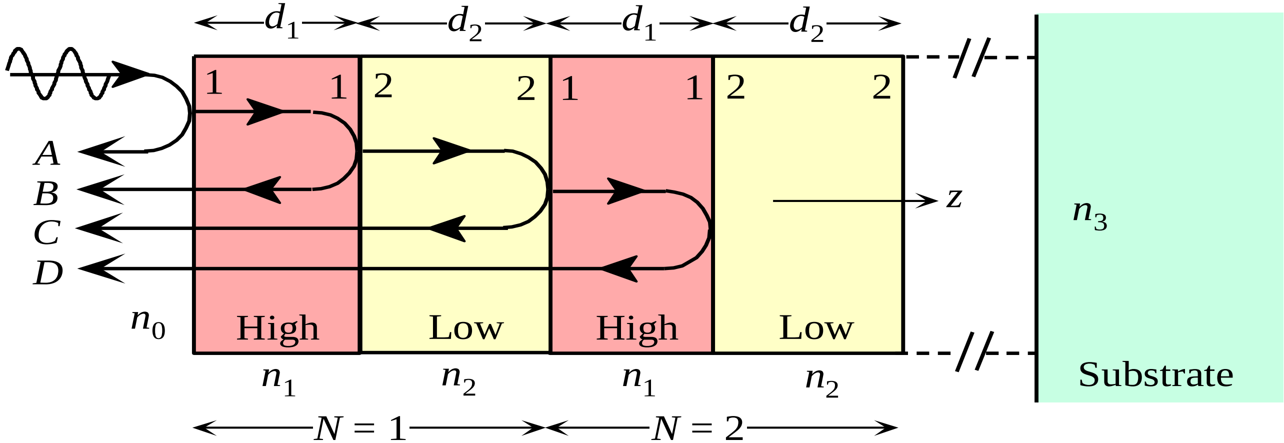
Layer thickness d = **Quarter of wavelength or λ_{layer}/ 4**

$\lambda_{layer} = \lambda_o/n$; λ_o is the free space wavelength at which the mirror is required to reflect the incident light, n = refractive index of layer.

Reflected waves from the interfaces **interfere constructively** and give rise to a substantial reflected light.

If there are **sufficient number** of layers, the reflectance can approach unity at λ_o.

Dielectric Mirror or Bragg Reflector



r_{12} for light in layer 1 being reflected at the 1-2 boundary is $r_{12} = (n_1 - n_2)/(n_1 + n_2)$ and is a **positive number** indicating **no phase change**.

r_{21} for light in layer 2 being reflected at the 2-1 boundary is $r_{21} = (n_2 - n_1)/(n_2 + n_1)$ which is $-r_{12}$ (negative) indicating **a π phase change**.

The reflection coefficient alternates in sign through the mirror

The **phase difference between *A* and *B*** is

$$0 + \pi + 2k_1 d_1 = 0 + \pi + 2(2\pi n_1/\lambda_o)(\lambda_o/4n_1) = 2\pi$$

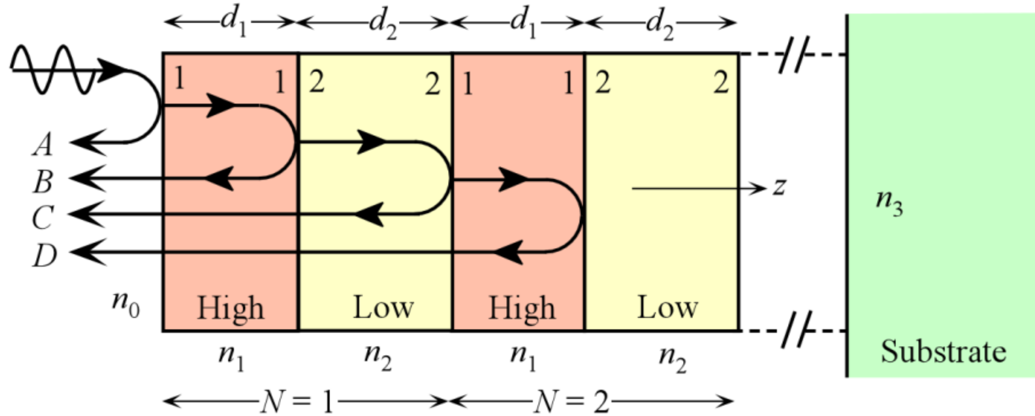
due to reflections at different boundaries

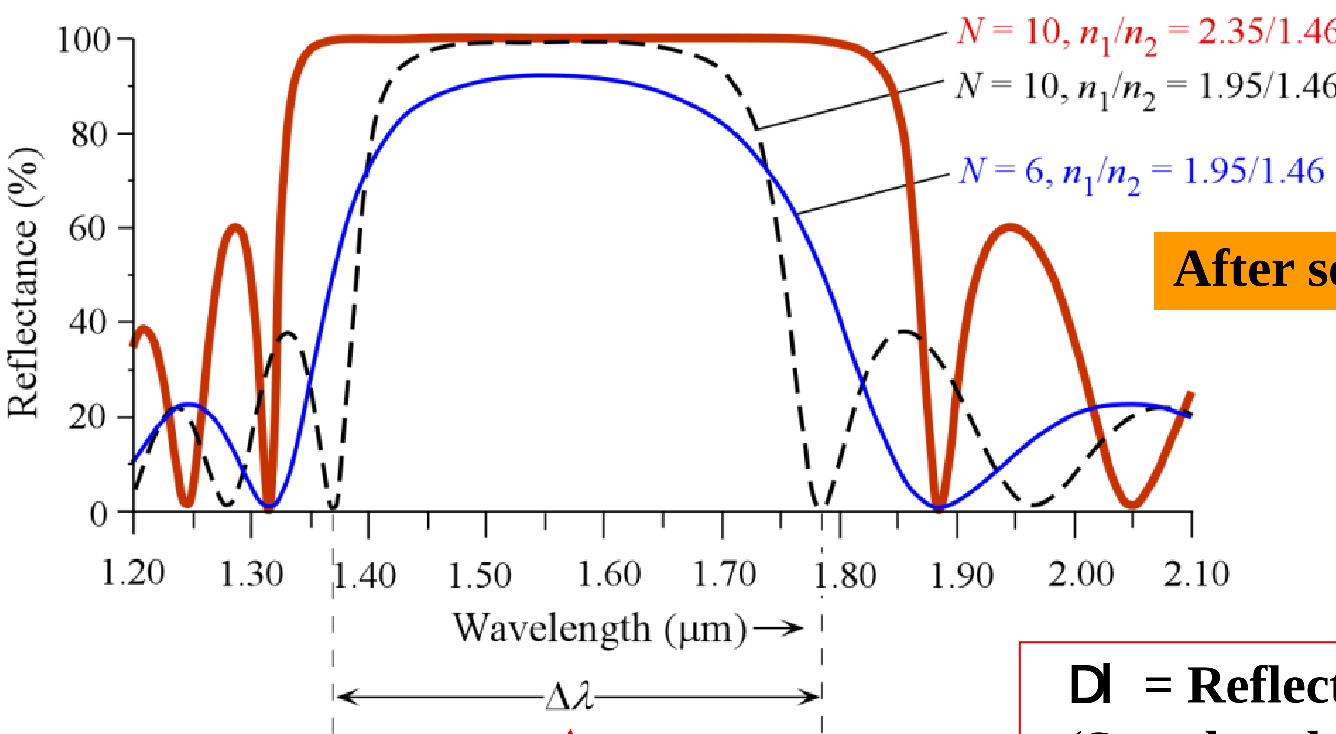
due to wave B travels an additional distance

Thus, waves *A* and *B* are in phase and **interfere constructively**.

Dielectric mirrors are widely used in modern vertical cavity surface emitting semiconductor lasers.

Dielectric Mirror or Bragg Reflector



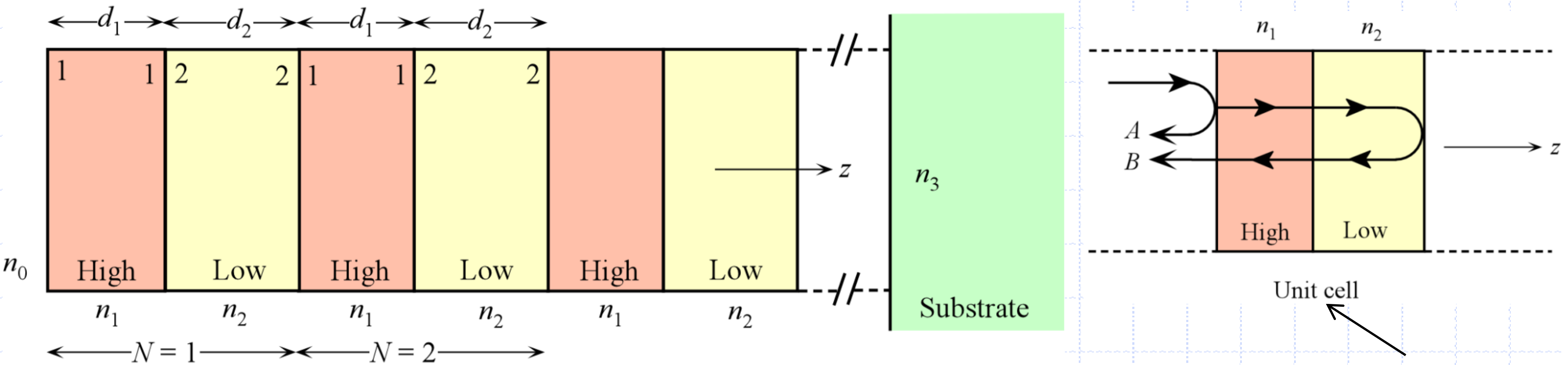


After several layers, $R \approx 1$.

Dl = Reflectance bandwidth
(**Stop-band** for transmittance)

Dielectric Mirror or Bragg Reflector

Consider an “infinite stack”



This is a “unit cell”

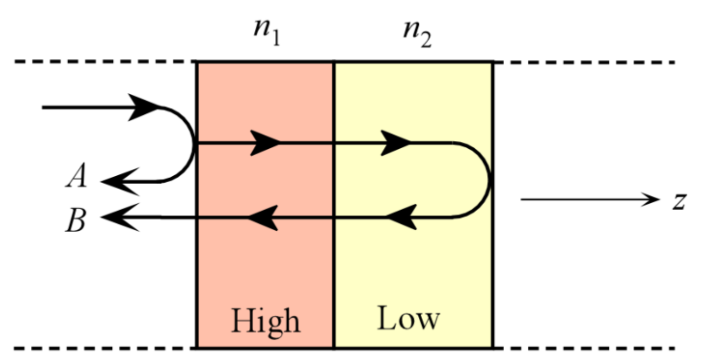
For reflection, the **phase difference between *A* and *B*** must be

$$2k_1d_1 + 2k_2d_2 = m(2\pi)$$

$$2(2pn_1/l\ )d_1 + 2(2pn_2/l\ )d_2 = m(2\pi)$$

$$n_1d_1 + n_2d_2 = m/\ /2$$

Dielectric Mirror or Bragg Reflector



Unit cell

$$n_1 d_1 + n_2 d_2 = l/2$$

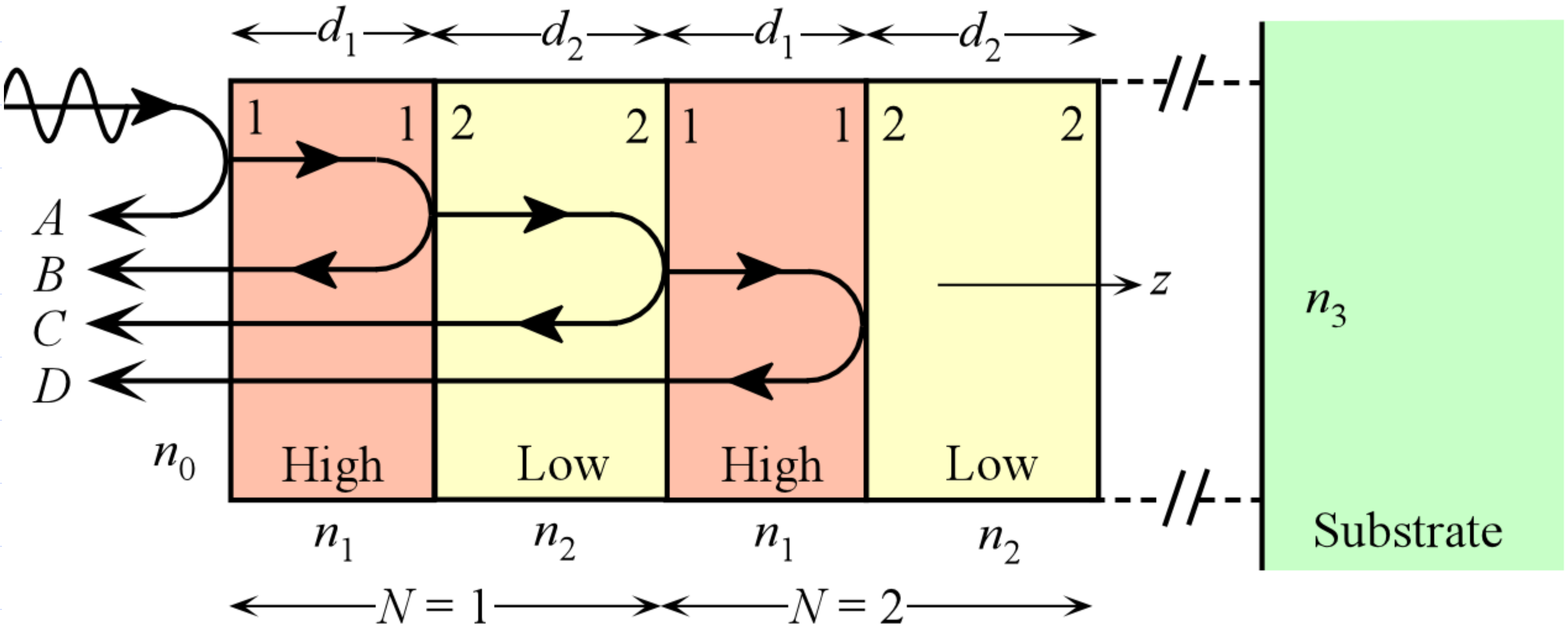
$$d_1 = l/4n_1$$

$$d_2 = l/4n_2$$

Quarter-Wave Stack

$d_1 = l/4n_1$ and $d_2 = l/4n_2$

Dielectric Mirror or Bragg Reflector



$$R_N = \left[\frac{n_1^{2N} - (n_0 / n_3) n_2^{2N}}{n_1^{2N} + (n_0 / n_3) n_2^{2N}} \right]^2$$

$$\frac{\Delta\lambda}{\lambda_o} \approx (4/\pi)\arcsin\left(\frac{n_1 - n_2}{n_1 + n_2}\right)$$

Example: Dielectric Mirror

A dielectric mirror has quarter wave layers consisting of Ta_2O_5 with n_H = **1.78** and SiO_2 with n_L = **1.55** both at **850 nm**, the central wavelength at which the mirror reflects light. The substrate is Pyrex glass with an index n_s = **1.47** and the outside medium is air with n_0 = **1**. Calculate the maximum reflectance of the mirror when the number N of double layers is **4** and **12**. What would happen if you use TiO_2 with n_H = **2.49**, instead of Ta_2O_5? Consider the N = **12** mirror. What is the bandwidth and what happens to the reflectance if you **interchange the high and low index layers**? Suppose we use a Si wafer as the substrate, what happens to the maximum reflectance?

Solution

$n_0 = 1$ for air, $n_1 = n_H = 1.78$, $n_2 = n_L = 1.55$, $n_3 = n_s = 1.47$, $N = 4$. For 4 pairs of layers, the maximum reflectance R_4 is

$$R_4 = \left[\frac{(1.78)^{2(4)} - (1/1.47)(1.55)^{2(4)}}{(1.78)^{2(4)} + (1/1.47)(1.55)^{2(4)}}\right]^2 = 0.4 \text{ or } 40\%$$

Solution

$N = 12$. For 12 pairs of layers, the maximum reflectance R_{12} is

$$R_{12} = \left[\frac{(1.78)^{2(12)} - (1/1.47)(1.55)^{2(12)}}{(1.78)^{2(12)} + (1/1.47)(1.55)^{2(12)}} \right]^2 = 0.906 \text{ or } 90.6\%$$

Now use TiO_2 for the high-n layer with $n_1 = n_H = 2.49$,
R_4 = 94.0% and **R_{12} = 100%** (to two decimal places).

The refractive index contrast is important. For the TiO_2-SiO_2 stack we only need 4 double layers to get roughly the same reflectance as from 12 pairs of layers of Ta_2O_5-SiO_2. If we **interchange n_H and n_L** in the 12-pair stack, *i.e.* $n_1 = n_L$ and $n_2 = n_H$, the Ta_2O_5-SiO_2 reflectance falls to 80.8% but the TiO_2-SiO_2 stack is unaffected since it is already reflecting nearly all the light.

Solution

We can only compare bandwidths $\Delta\lambda$ for "infinite" stacks (those with $R \approx 100\%$) For the TiO_2-SiO_2 stack

$$\Delta\lambda \approx \lambda_o (4/\pi) \arcsin\left(\frac{n_2 - n_1}{n_2 + n_1}\right)$$

$$\Delta\lambda \approx (850 \text{ nm})(4/\pi) \arcsin\left(\frac{2.49 - 1.55}{2.49 + 1.55}\right) = 254\text{nm}$$

For the Ta_2O_5-SiO_2 infinite stack, we get D/ =74.8 nm

As expected D/ is narrower for the smaller contrast stack

MAGNETIC FIELD, IRRADIANCE AND POYNTING VECTOR

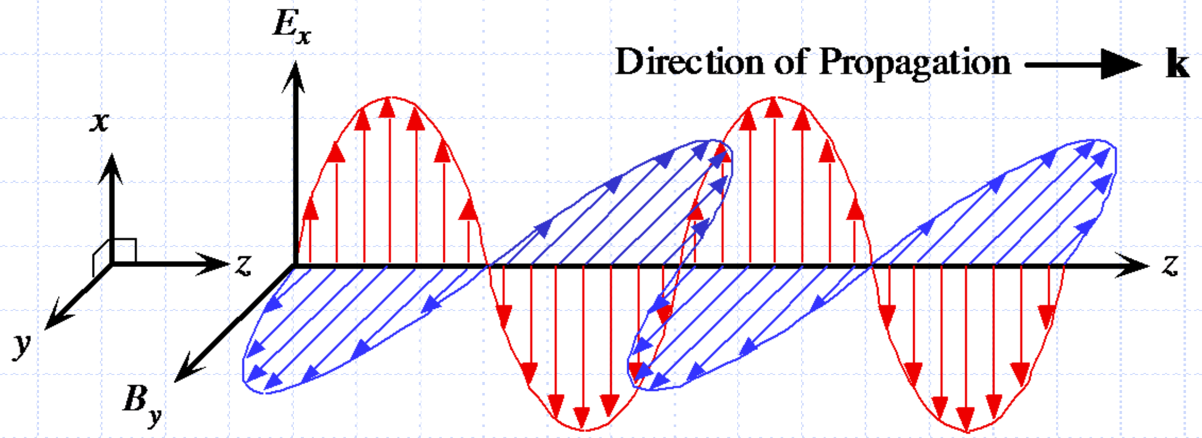
Magnetic field

- The magnetic filed (magnetic induction) component B_y always accompanies E_x in an EM wave propagation.
- For an EM wave in an isotropic dielectric medium

$$E_x = vB_y = \frac{c}{n} B_y$$

$$\text{where } v = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0}} \quad \text{phase velocity}$$

$$n = \sqrt{\varepsilon_r} \quad \text{refractive index}$$



Energy density

- Energy density (energy per unit volume) in the E_x field

$$u_E = \frac{1}{2}\varepsilon_0\varepsilon_r E_x^2$$

- Energy density in the B_y field

$$u_M = \frac{1}{2\mu_0} B_y^2$$

- The energy densities are the same

$$\frac{1}{2}\varepsilon_0\varepsilon_r E_x^2 = \frac{1}{2}\varepsilon_0\varepsilon_r (vB_y)^2 = \frac{1}{2}\varepsilon_0\varepsilon_r \left(\frac{1}{\sqrt{\varepsilon_0\varepsilon_r\mu_0}} B_y\right)^2 = \frac{1}{2}\frac{1}{\mu_0} By^2$$

- Total energy density in the wave

$$u_{EM} = \varepsilon_0\varepsilon_r E_x^2$$

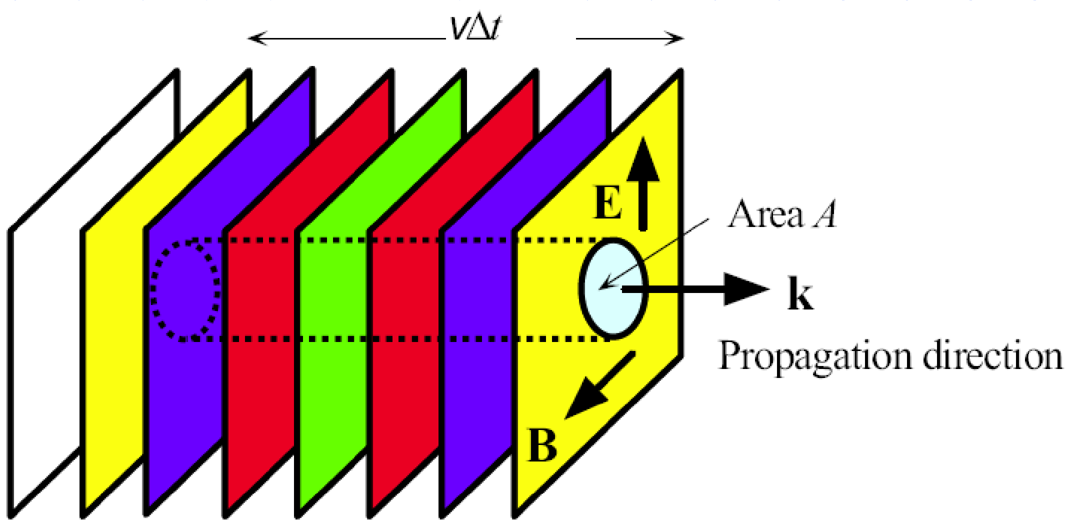
Poynting Vector and EM Power Flow

- As the EM wave propagates in the direction of **k**, there is an energy flow in this direction. The wave brings with it EM energy.
- If S is the EM power flow per unit area, then

$$S = \text{Energy flow per unit time per unit area}$$

$E_x = vB_y$

$$\Rightarrow S = \frac{(Av\Delta t)(\varepsilon_0\varepsilon_r E_x^2)}{A\Delta t} = v\varepsilon_0\varepsilon_r E_x^2 = v^2\varepsilon_0\varepsilon_r E_x B_y$$



A plane EM wave traveling along **k** crosses an area A at right angles to the direction of propagation. In time Δt, the energy in the cylindrical volume $Av\Delta t$ (shown dashed) flows through A.

Poynting Vector and Irradiance

- In an isotropic medium, the energy flow is in the direction of wave propagation. If we use the vectors **E** and **B** to represent the electric and magnetic fields in the EM wave, then **E**×**B** is in a direction of wave propagation.
- The EM power flow per unit area:

$$\mathbf{S} = v^2 \varepsilon_o \varepsilon_r \mathbf{E}\times\mathbf{B}$$

- **S,** called the **Poyntin vector**, represents the energy flow per unit time per unit area in a direction determined by **E**×**B** (direction of propagation).
- Its magnitude, power flow per unit area, is called the **irradiance (instantaneous irradiance, or intensity)**.
- $S = |v^2 \varepsilon_0 \varepsilon_r \mathbf{E}\times\mathbf{B}| = v^2 \varepsilon_0 \varepsilon_r E_x B_y = v \varepsilon_0 \varepsilon_r E_x^{\,2}$.

Average Irradiance (Intensity)

If $E_x = E_0 \sin(\omega t)$

$$\langle E_x^2 \rangle = \frac{1}{T}\int_0^T \left(E_0 \sin(\omega t)\right)^2 dt = \frac{1}{2}E_0^2$$

$$S = v\varepsilon_0\varepsilon_r E_x^2 \qquad \Rightarrow \quad I = S_{\text{average}} = \tfrac{1}{2} v\varepsilon_o\varepsilon_r E_o^2$$

- S_{average} is called the **average irradiance (intensity)**.

$$I = S_{\text{average}} \propto E_0^2$$

Since $v = c/n$ and $\varepsilon_r = n^2$ we can write

$$c = 3\times10^8 \text{ m/s}, \ \varepsilon_0 = 8.85419\times10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2,$$

$$\Rightarrow \frac{1}{2}c\varepsilon_0 = 1.328\times10^{-3} \text{ C}^2/\text{J}\cdot s,$$

$$\Rightarrow I = S_{\text{average}} = (1.33\times10^{-3})nE_0^2 \quad \text{Watt/m}^2$$

in N/C (or V/m) (annotation on E_0)

Instantaneous Irradiance and Average Irradiance

$$S = v\varepsilon_0\varepsilon_r E_x^2 = v\varepsilon_0\varepsilon_r E_0^2 \cos^2(\omega t) \quad \text{(instantaneous irradiance)}$$

$$I = S_{\text{average}} = \frac{1}{2} v\varepsilon_0\varepsilon_r E_0^2 \quad \text{(average irradiance)}$$

- The instantaneous irradiance can only be measured if the power meter can respond more quickly than the oscillations of the electric field.
- Optical frequency ~ 10^{14} Hz.

□All detectors have a response rate much slower than the frequency of the wave.

□All practical measurements yield the average irradiance S_{average}.

EXAMPLE: Electric and Magnetic Fields in Light

- Laser beam from a He-Ne laser
  Intensity (average irradiance) $I = 1\ \text{mW/cm}^2$
- $E_0 = ?$, $B_0 = ?$ in air
- $E_0 = ?$, $B_0 = ?$ in a glass (n = 1.45)

- ***Solution***
  The intensity (average irradiance)

$$I = S_{\text{average}} = \frac{1}{2} c\varepsilon_0 n E_0^2 \Rightarrow E_0 = \sqrt{\frac{2I}{c\varepsilon_0 n}}$$

EXAMPLE: Electric and Magnetic Fields in Light

- In air, $n = 1$

$$E_0 = \sqrt{\frac{2\times(1\times10^{-3}\times10^{4}\ \text{Wm}^{-2})}{(3\times10^{8}\ \text{m/s})(8.85\times10^{-12}\ \text{Fm}^{-1})(1)}}$$

$$= 87\ \text{Vm}^{-1}\ \text{or}\ 0.87\ \text{Vcm}^{-1}$$

$$B_0 = E_0/c = (87\ \text{Vm}^{-1})/(3\times10^{8}\ \text{m/s}) = 0.29\,\mu\text{T}$$

$$(\text{recall}\ E_0 = vB_0)$$

- In a glass medium, $n = 1.45$

$$E_0(\text{medium}) = \sqrt{\frac{2\times(1\times10^{-3}\times10^{4}\ \text{Wm}^{-2})}{(3\times10^{8}\ \text{m/s})(8.85^{-2}\times10^{-12}\ \text{Fm}^{-1})(1.45)}}$$

$$= 72\ \text{Vm}^{-1}$$

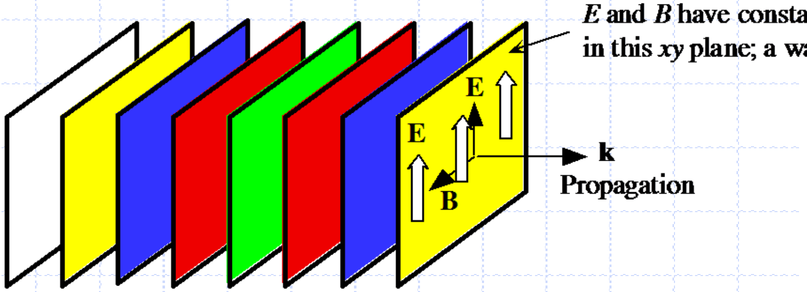
$$B_0(\text{medium}) = E_0/v = nE_0/c = (1.45)(72\ \text{Vm}^{-1})/(3\times10^{8}\ \text{m/s})$$

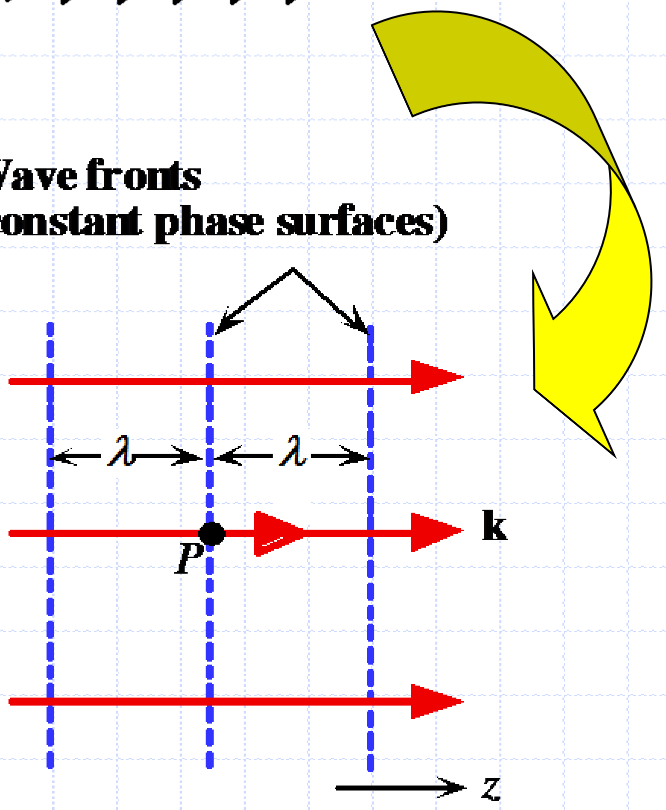
$$= 0.35\,\mu\text{T}$$

Maxwell's Wave Equation and Diverging Waves

Plane Wave is an Idealization

- The propagation vectors everywhere are all parallel and the plane wave propagates without the wave diverging. ***(plane wave has no divergence)***.
- Amplitude E_0 is the same at all points on a given plane perpendicular to **k**.
- Planes extend to ∞ **energy $\rightarrow \infty$**
- We need an infinite large EM source to generate a perfect plane wave!
- In reality, the E in a plane at right angles to **k** does not extend to ∞ since the light beam would a **finite cross sectional area** and finite power.
- **A plane is an idealization** that is useful in analyzing many phenomena.





A perfect plane wave

Possible waves that satisfy Maxwell's EM wave equation

$$\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial z^2}=\varepsilon_o\varepsilon_r\mu_o\frac{\partial^2 E}{\partial t^2}$$

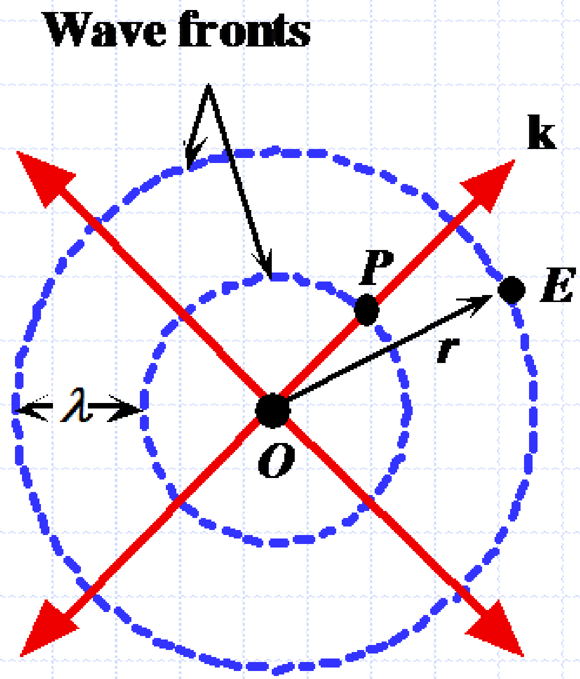
- To find the time and space dependence of the field, we must solve Maxwell's wave equation. in conjunction with the **initial and boundary conditions**.
- There are many possible waves that satisfy Maxwell's wave equation:
  - Plane wave
  - Spherical wave
  - Cylindrical wave
  - …

Spherical wave

- A spherical wave is described by a traveling field that emerges from a **point EM source**:

$$E = \frac{A}{r}\cos(\omega t - kr) \qquad (6)$$

A: a constant.



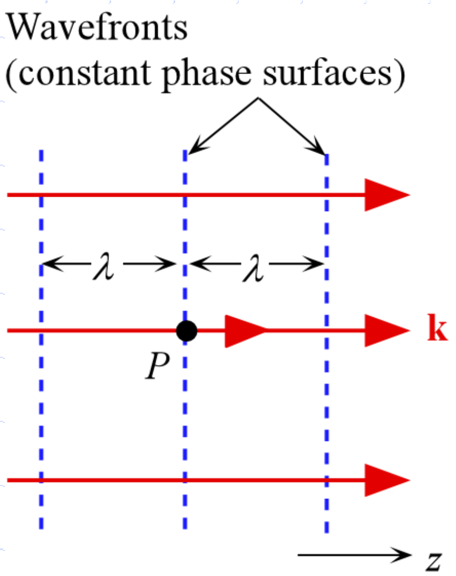
A perfect spherical wave

- The **wavefronts are spheres** centered at the point source *O*.
- Amplitude decays with distance *r* from the source:

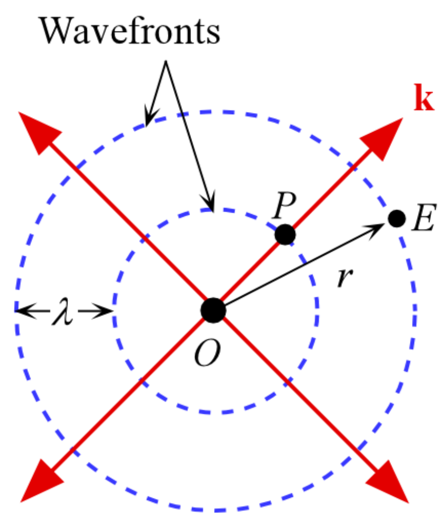
$$A/r \xrightarrow[r\to\infty]{} 0$$

- **k wavevectors diverge out** and, as the wave propagates, **the constant surfaces becomes larger**.

Optical Divergence



A perfect plane wave

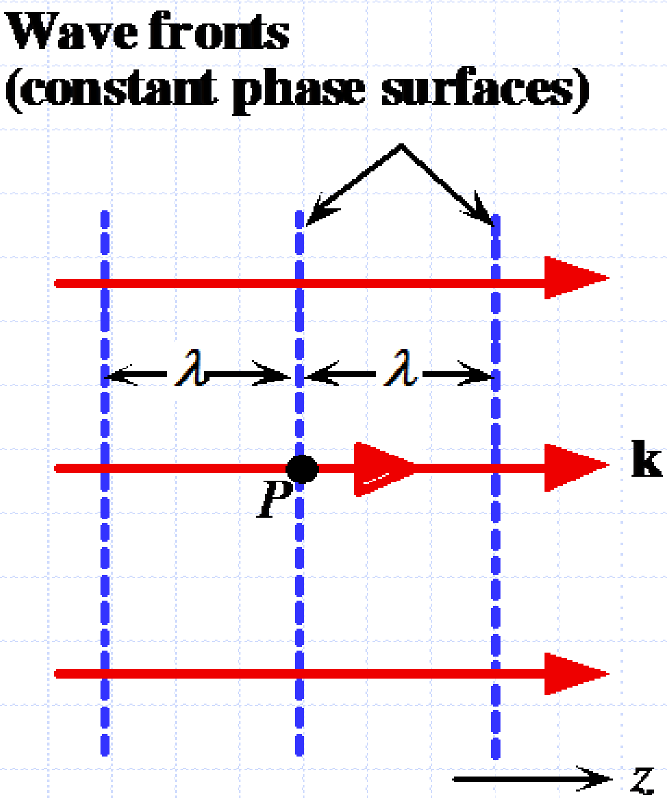


A perfect spherical wave

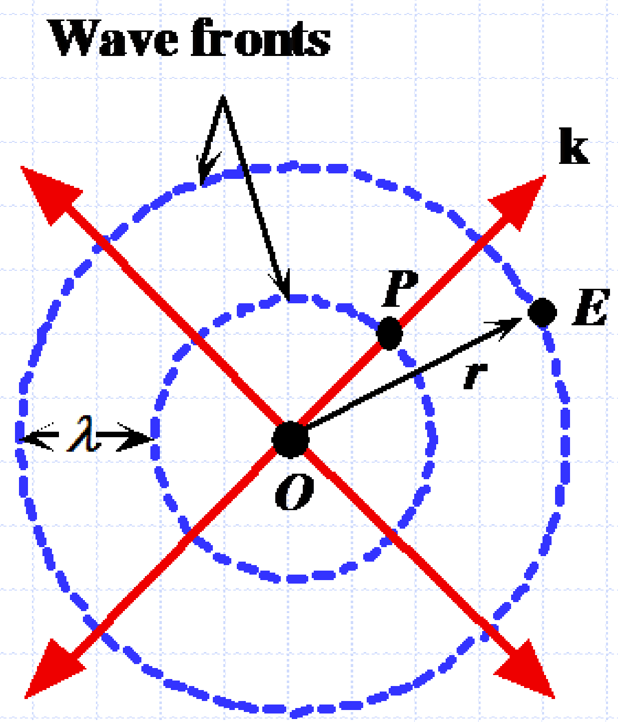
- **Optical divergence**
  - refers to the angular separation of wavevectors on a given wavefront.
- Spherical wave
  - 360° divergence (fully diverging wavevectors)
- Plane wave
  - 0° divergence (perfectly parallel wave vectors)

Waves from ideal EM sources

- Infinitely large source produces plane wave
- Point source produces spherical wave



A perfect plane wave

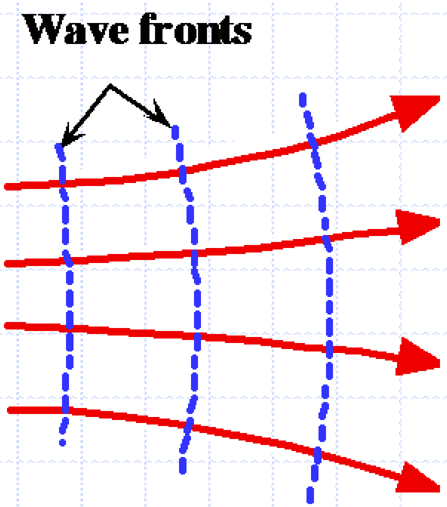


A perfect spherical wave

Wave from a practical EM source

- In reality, an EM source would have a **finite size and finite power.**

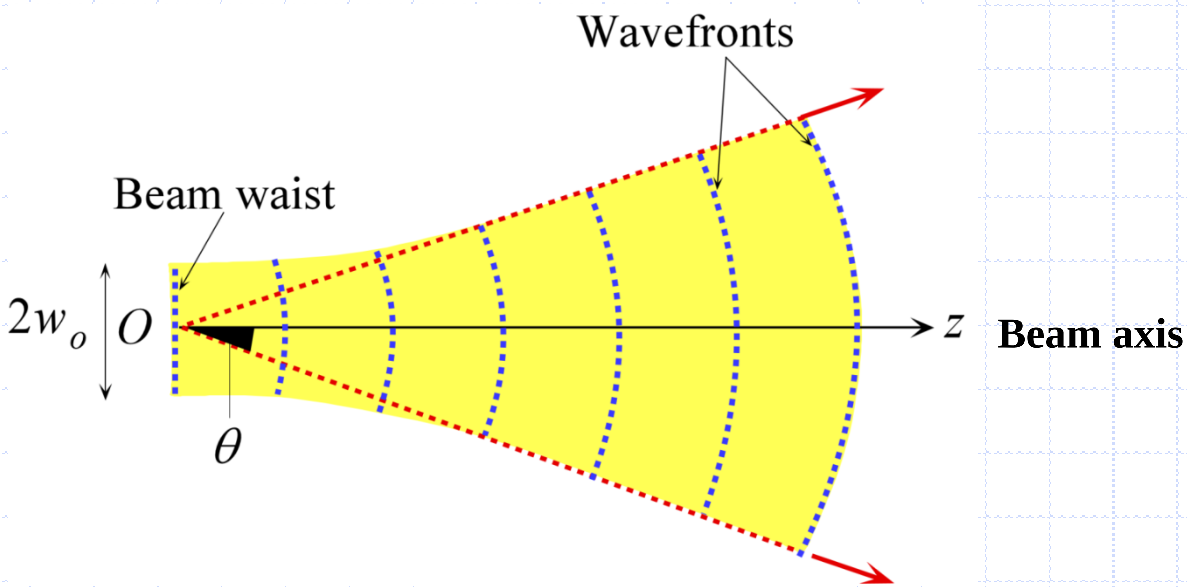
- The light beam exhibits some **inevitable divergence** while propagating.
- The **wavefronts are slowly bent away** thereby spreading the wave.
- Light rays **slowly diverge away** from each other.



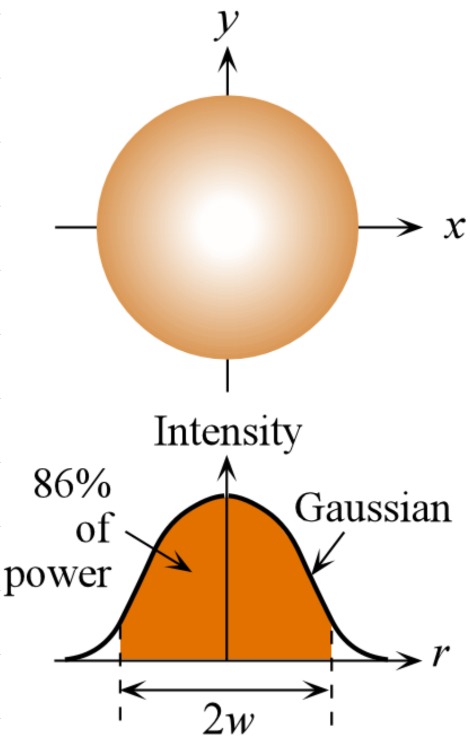
A divergent beam

Gaussian beams

- Many light beams can be described by assuming that they are **Gaussian beams**.
  - Ex. The output from a **laser**
- It is a result of radiation **from a source of finite extent**
- Properties:
  - Still **exp *j*(ω*t* – *kz*) dependence**
  - **Amplitude varies spatially** away from the axis and also along the axis.
  - **Slow diverges**
  - **Intensity distribution** across the beam cross-section **is Gaussian**.



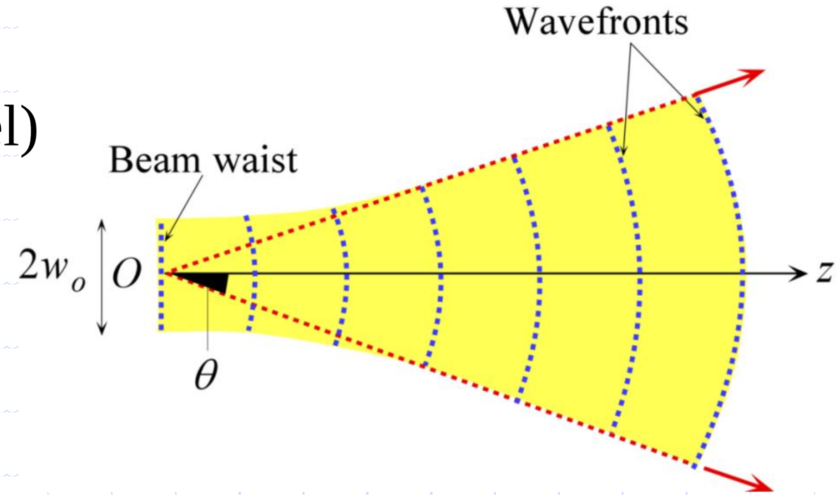
Wavefronts of a Gaussian light beam



$$I(r, z) = I_o \exp(-2r^2/w^2)$$

Gaussian beam

- It starts from O with a finite width $2w_0$ where the wavefronts are parallel and then the beam slowly diverges as the wavefronts curve out during propagation along z.
  - waist: $2w_0$ (where the wavefronts are parallel)
  - waist radius: w_0
  - spot size: $2w_0$
  - beam divergence: 2θ



- The increase in beam diameter $2w$ with z makes an angles 2θ at O which is called the **beam divergence**.

$$2\theta = \frac{4\lambda}{\pi(2w_0)} \quad \Rightarrow \quad \theta \propto \frac{1}{w_0}$$

- The greater the waist, the narrower the divergence.

EXAMPLE: A diverging laser beam

- Consider a He-Ne laser beam at 633 nm with spot size of 1 mm. Assuming a Gaussian beam, what is the divergence of the beam?

Solution

- HeNe laser
  - $\lambda = 633 \text{ nm} = 633 \times 10^{-9} \text{ m}$
- Spot size $2w_0 = 1 \text{ mm} = 1 \times 10^{-3} \text{ m}$

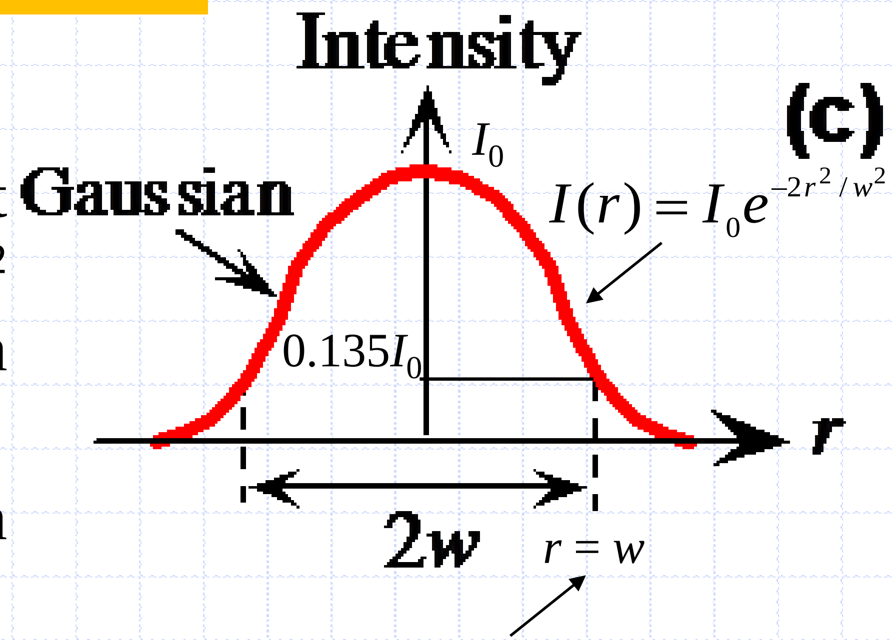
$$2\theta = \frac{4\lambda}{\pi(2w_0)} = \frac{4 \times (633 \times 10^{-9}\,\text{m})}{\pi \times (1 \times 10^{-3}\,\text{m})}$$

$$= 8.06 \times 10^{-5}\,\text{rad} = 0.0046^\circ$$

Gaussian distribution

$$I(r) = I_o \exp[-2(r/w)^2]$$

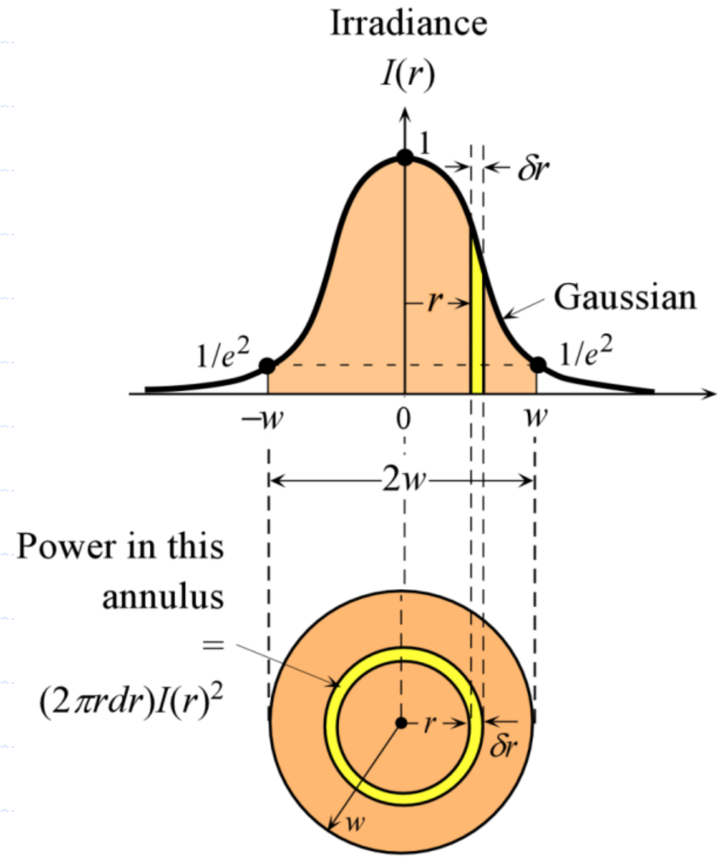
- **Beam diameter** $2w$
  - It is defined in such way that the cross sectional area πw^2 contains 85% of the beam power.
  - It increases as the beam traveling along z.



$$I(r = w) = I_0 e^{-2w^2/w^2}$$
$$= I_0 e^{-2} = 0.135 I_0$$

Power contained in the area of πw^2

Area of a circular thin strip (annulus) with radius r is $2\rho rdr$. Power passing through this strip is proportional to $I(r)\,(2\rho r)dr$



- Total power

$$P_{total} = \int_0^\infty \int_0^{2\pi} I(r) r dr d\theta$$

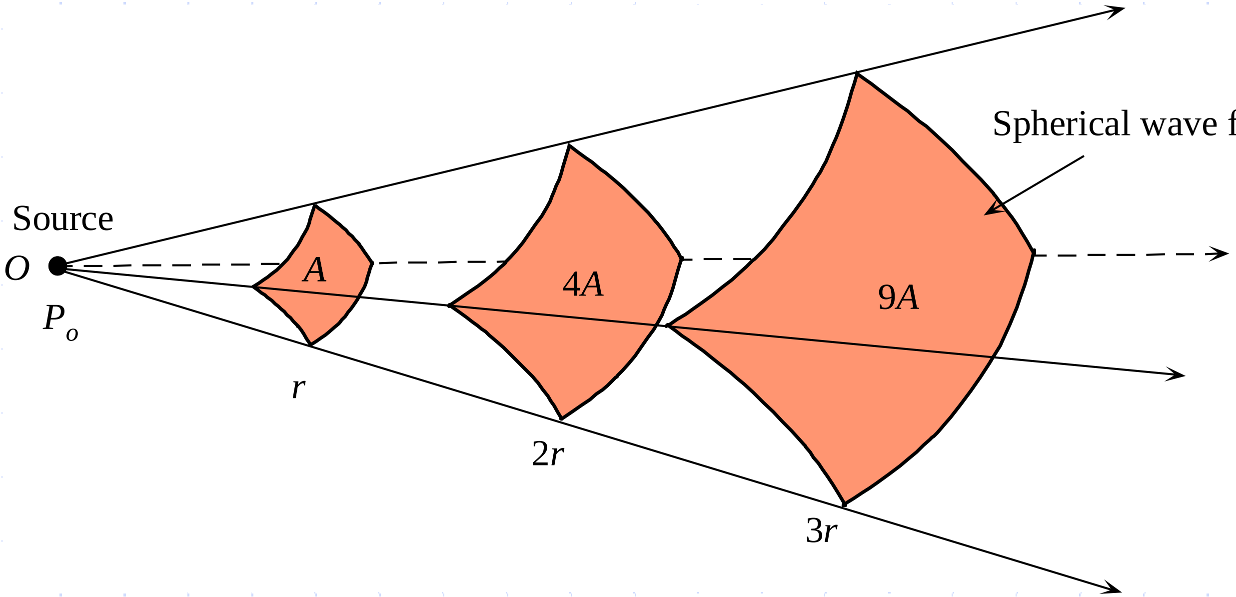
$$= 2\pi \int_0^\infty I_0 e^{-2r^2/w^2} r dr = \frac{1}{2}\pi w^2 I_0$$

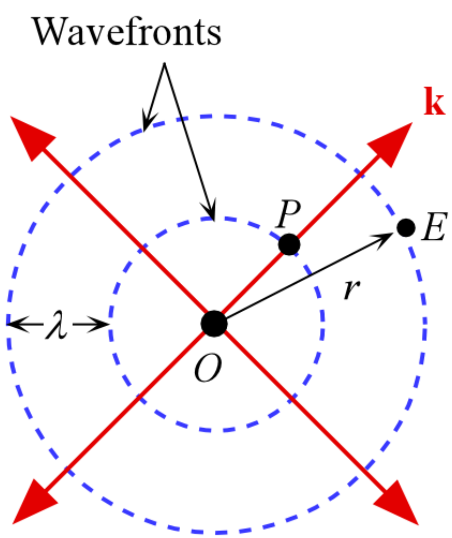
- Power contained in the area of πw^2

$$P_{area} = \int_0^w \int_0^{2\pi} I(r) r dr d\theta = 2\pi \int_0^w I_0 e^{-2r^2/w^2} r dr = \frac{1}{2}\pi w^2 I_0 (1 - e^{-2})$$

$$\Rightarrow \frac{P_{area}}{P_{total}} = (1 - e^{-2}) = 0.865$$

Irradiance of a Spherical Wave



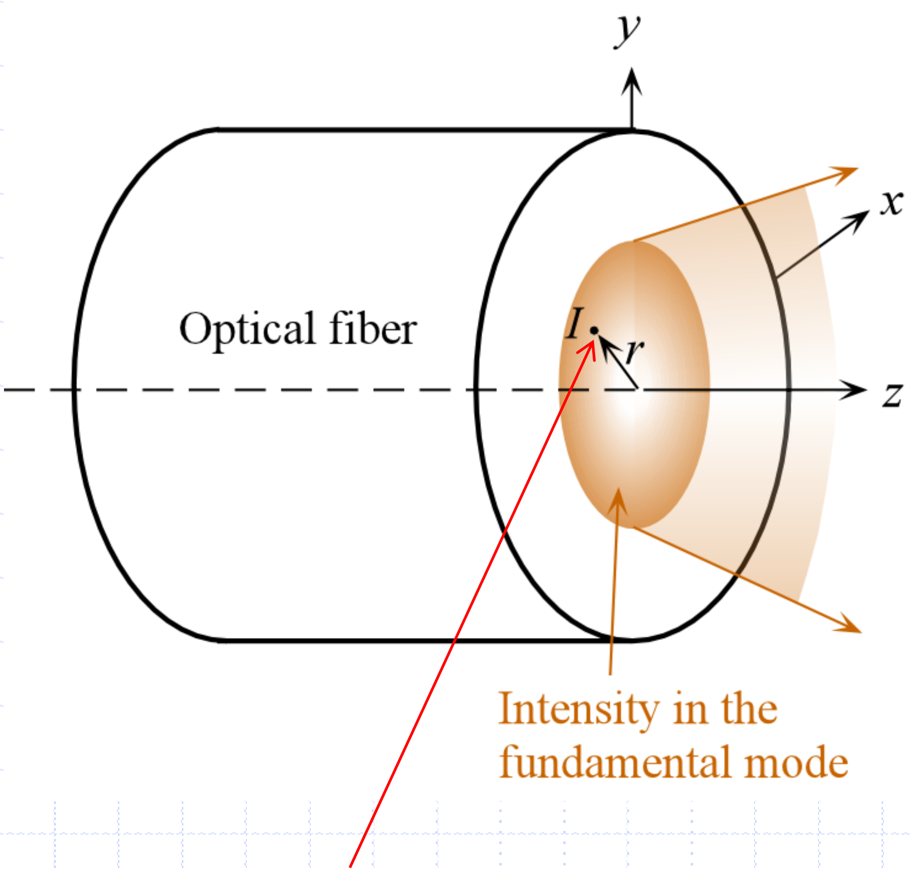


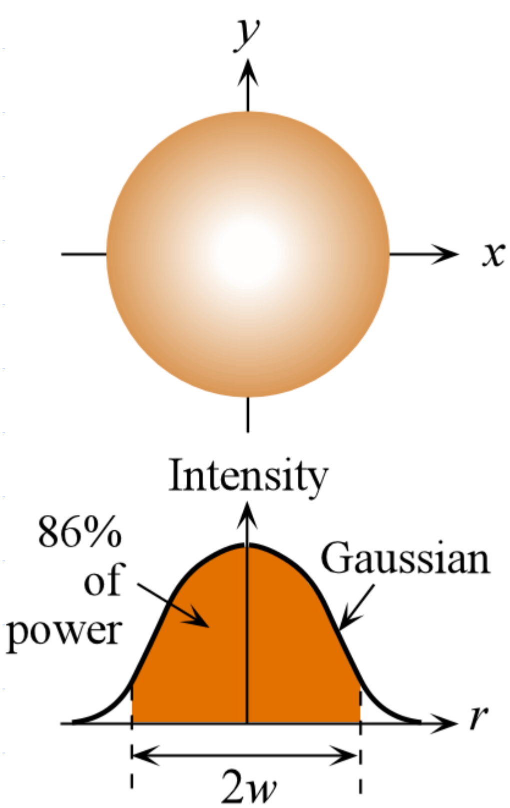
Perfect spherical wave

$$I = \frac{P_o}{4\pi r^2}$$

The Gaussian Intensity Distribution is Not Unusual

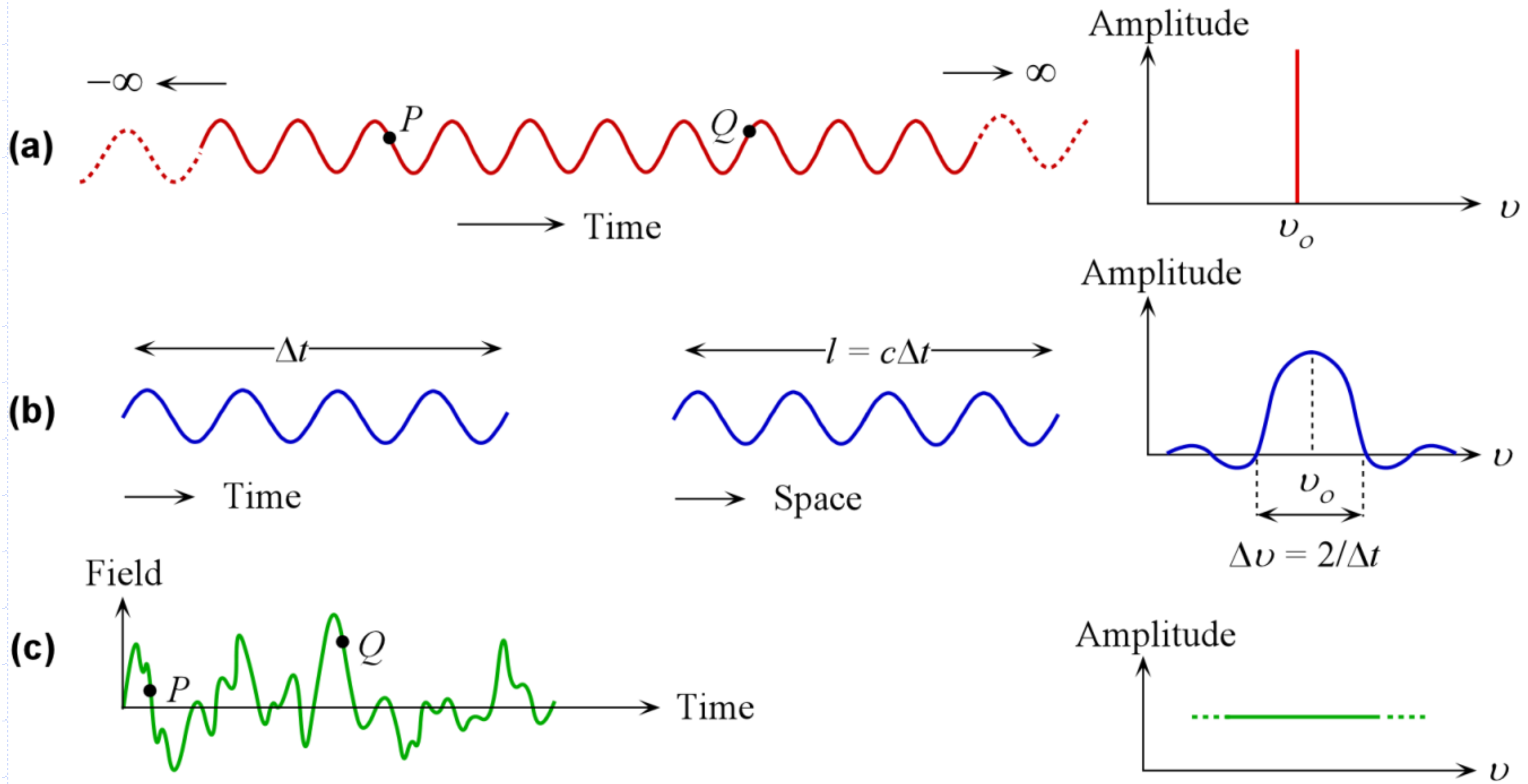
The Gaussian intensity distribution is also used in fiber optics
The fundamental mode in single mode fibers can be approximated with a Gaussian intensity distribution across the fiber core





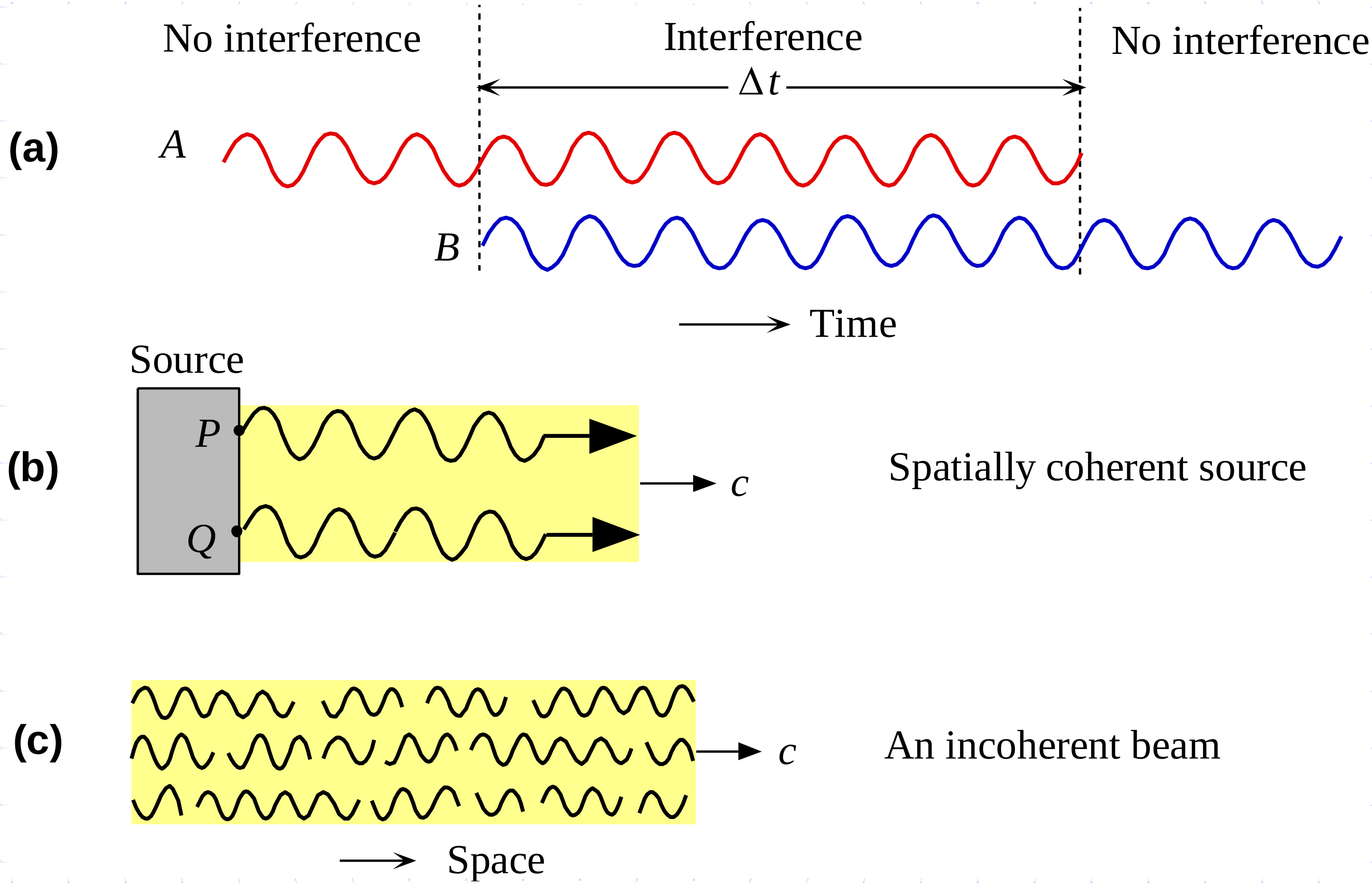
$$I(r) = I_o \exp(-2r^2/w^2)$$

Temporal and Spatial Coherence



(a) A sine wave is perfectly coherent and contains a well-defined frequency υ_o. (b) A finite wave train lasts for a duration D*t* and has a length *l*. Its frequency spectrum extends over Du = 2/D*t*. It has a coherence time D*t* and a coherence length *l*. (c) White light exhibits practically no coherence.

Temporal and Spatial Coherence



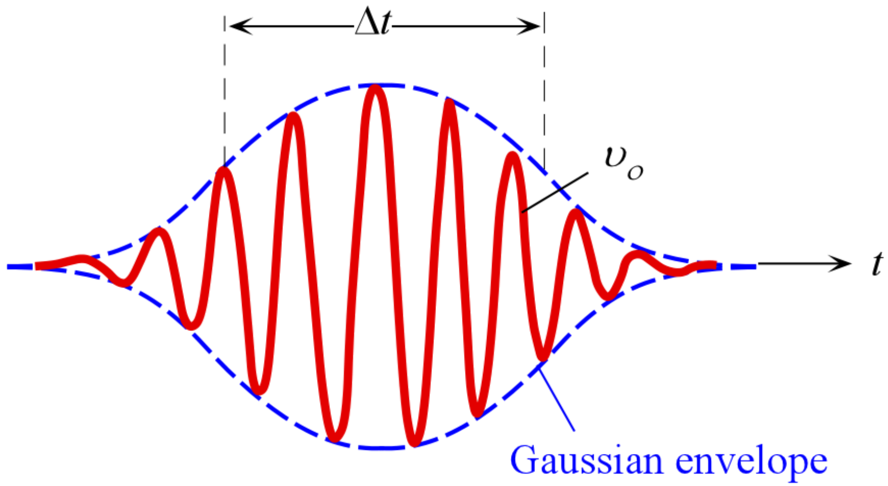
(a) Two waves can only interfere over the time interval D*t*. (b) Spatial coherence involves comparing the coherence of waves emitted from different locations on the source. (c) An incoherent beam

Temporal and Spatial Coherence

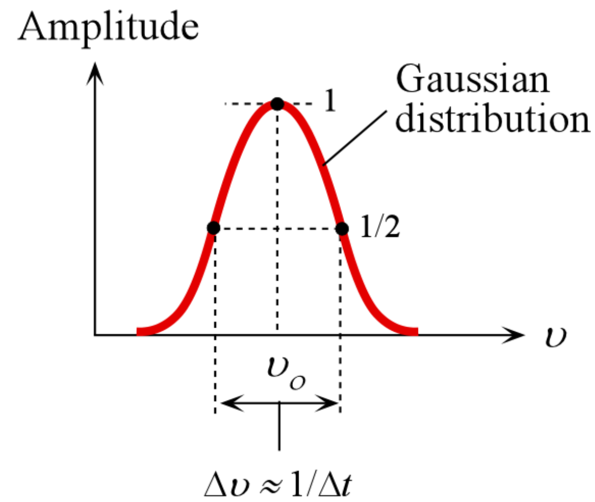
Dt = coherence time

$l = cDt$ = coherence length

For a Gaussian light pulse



Gaussian wave packet



Spectrum

$$\Delta \upsilon \approx \frac{1}{\Delta t}$$

FWHM spreads

Spectral width

Pulse duration

Temporal and Spatial Coherence

Dt = **coherence time**

$l = cDt$ = **coherence length**

$$\Delta \upsilon \approx \frac{1}{\Delta t}$$

Na lamp, orange radiation at 589 nm has spectral width $Du \approx 5 \times 10^{11}$ Hz.

$Dt \approx 1/\,Du = 2 \times 10^{-12}$ s or 2 ps,

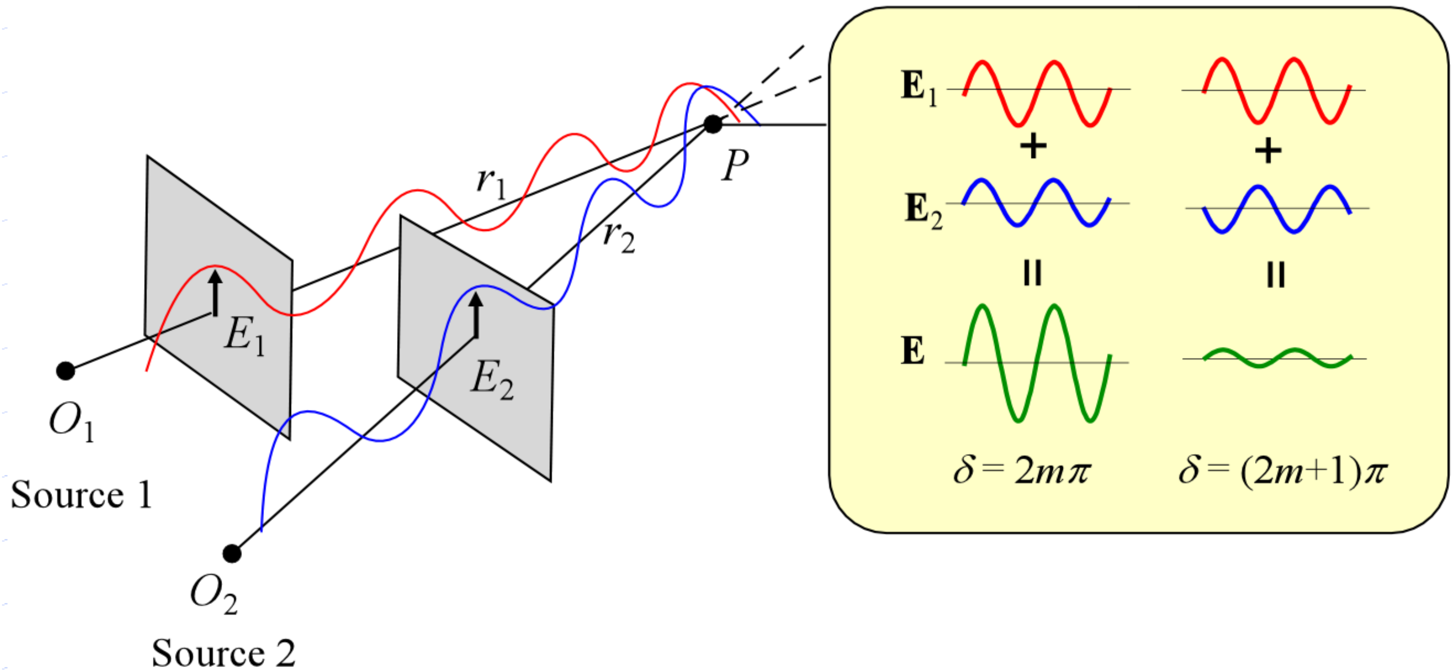
and its coherence length $l = cDt$,

$l = 6 \times 10^{-4}$ m or 0.60 mm.

He-Ne laser operating in multimode has a spectral width around 1.5×10^{9} Hz, $Dt \approx 1/Du = 1/(1.5 \times 10^{9})$ s or 0.67 ns

$l = cDt = 0.20$ m or 200 mm.

Interference

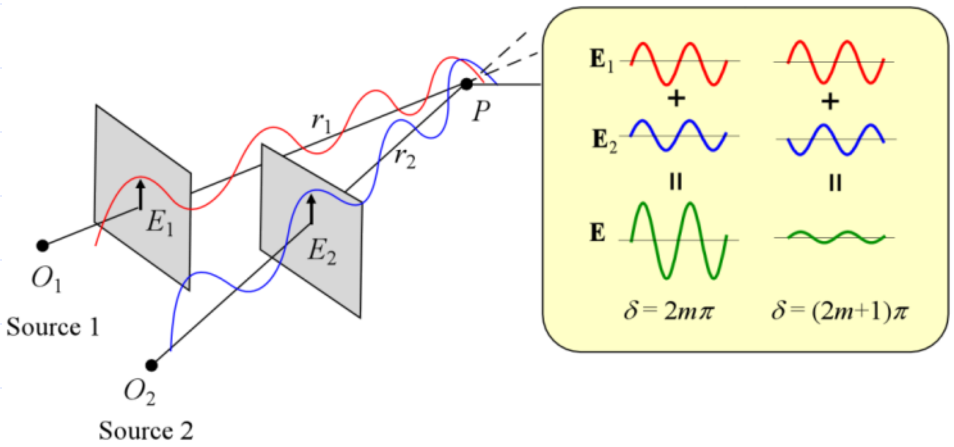


$\mathbf{E}_1 = \mathbf{E}_{o1}\sin(\omega t - kr_1 - \phi_1)$ and $\mathbf{E}_2 = \mathbf{E}_{o2}\sin(\omega t - kr_2 - \phi_2)$

Interference results in $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$

$$\overline{\mathbf{E}\cdot\mathbf{E}} = \overline{(\mathbf{E}_1 + \mathbf{E}_2)\cdot(\mathbf{E}_1 + \mathbf{E}_2)} = \overline{\mathbf{E}_1^2} + \overline{\mathbf{E}_2^2} + 2\overline{\mathbf{E}_1\mathbf{E}_2}$$

Interference



Resultant intensity I is

$$I = I_1 + I_2 + 2(I_1 I_2)^{1/2}\cos d$$

$$d = k(r_2 - r_1) + (f_2 - f_1)$$

Phase difference due to optical path difference

Constructive interference and **Destructive interference**

$$I_{max} = I_1 + I_2 + 2(I_1 I_2)^{1/2}$$

and

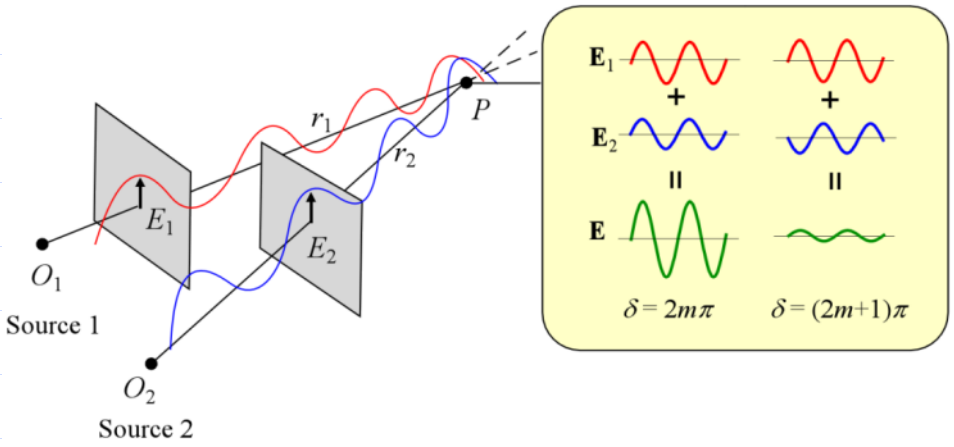
$$I_{min} = I_1 + I_2 - 2(I_1 I_2)^{1/2}$$

If the interfering beams have equal irradiances, then

$$I_{max} = 4I_1$$

$$I_{min} = 0$$

Interference between coherent waves



Resultant intensity I is

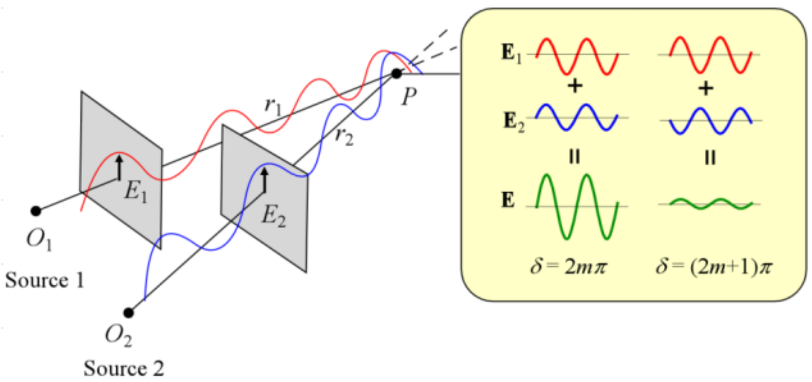
$$I = I_1 + I_2 + 2(I_1 I_2)^{1/2} \cos d$$

$$d = k(r_2 - r_1) + (f_2 - f_1)$$

Interference between incoherent waves

$$I = I_1 + I_2$$

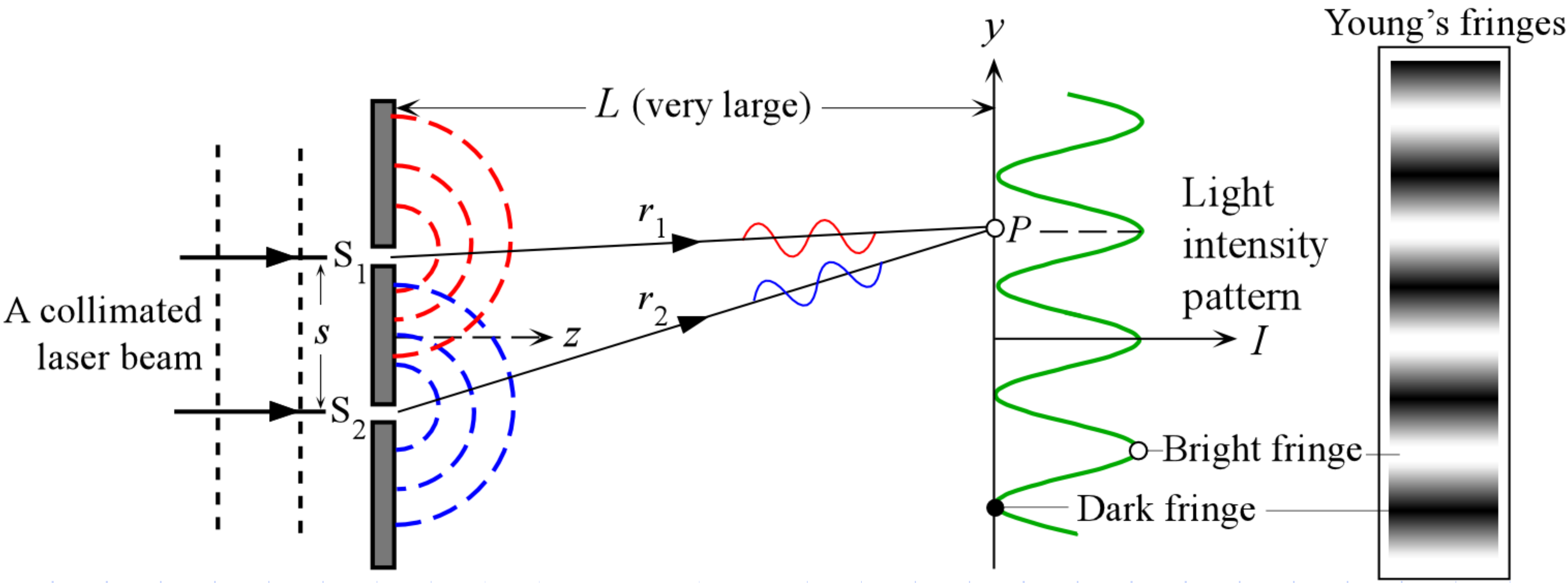
Interference between coherent waves



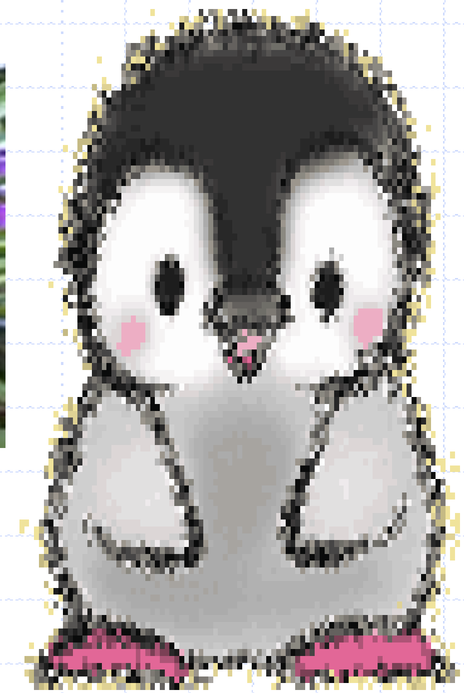
Resultant intensity *I* is

$$I = I_1 + I_2 + 2(I_1 I_2)^{1/2}\cos d$$

$$d = k(r_2 - r_1) + (f_2 - f_1)$$



Thank you



Have a nice day.