Fall 2019: Classical Field Theory (PH6297)

Scalar Field Theory III: Complex Scalar, Symmetries & Noether's theorem*

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1 Complex Scalar field theory

A quick generalization of the free real scalar field theory is obtained by complexifying the field. We will denote the complex field by upper case Greek symbol, $\Phi(x)$. The main contrast with the real scalar field theory is that the complex scalar field theory will admit an **internal** symmetry called the global $U(1)$ symmetry which we will see via Noether's theorem to lead to a conserved charge. Further we will identify this charge as the electric charge when we couple the complex scalar to a Maxwell gauge field, A_μ. As usual, the very first step in the study of any physical system, here in this case the complex scalar field theory, is writing down the action functional. Since the action must be real, the action is constrained to be the following,

$$I\left[\Phi(x), \Phi^\dagger(x)\right] = \int d^4x \left[\left(\partial_\mu \Phi\right)^\dagger \partial^\mu \Phi - V\left(\Phi^\dagger \Phi\right)\right]. \tag{1}$$

Here $\Phi^\dagger$ is the complex conjugate of Φ. For simplicity we take, $V\left(\Phi^\dagger\Phi\right) = m^2 \Phi^\dagger \Phi$, which as expected will give rise to a free theory i.e. one with equations of motion linear in Φ or $\Phi^\dagger$. The classical equation of motion for the complex field theory are,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi^\dagger\right)}\right) - \frac{\partial \mathcal{L}}{\partial \Phi^\dagger}.$$

Plugging the expression for $\mathcal{L}$ from (1) is same as that of the real scalar field, i.e. the Klein-Gordon equation,

$$\left(\partial^2 + m^2\right)\Phi = \left(\partial^2 + m^2\right)\Phi^\dagger = 0.$$

Upon rewriting this complex scalar field into it's real and imaginary components,

$$\Phi = \frac{\phi_1 + i\,\phi_2}{\sqrt{2}}, \Phi^\dagger = \frac{\phi_1 - i\,\phi_2}{\sqrt{2}},$$

we find out that the complex scalar field theory is a theory of two **non-interacting** real scalar field theories. This is because on splitting the field into its real and imaginary components, the action splits into two pieces as well,

$$\begin{aligned} I\left[\Phi(x), \Phi^\dagger(x)\right] &= \int d^4x \left[\left(\partial_\mu \Phi\right)^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi\right], \\ &= \int d^4x \left(\frac{1}{2}\partial_\mu \phi_1\, \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2\right) + \int d^4x \left(\frac{1}{2}\partial_\mu \phi_2\, \partial^\mu \phi_2 - \frac{1}{2} m^2 \phi_2^2\right). \end{aligned}$$

*Notes for Lecture 12 (Aug. 23, 2019)

1.1 Global $U(1)$ Symmetry of the complex field theory

One can easily check that the complex scalar field theory action Eq. (1) is invariant under multiplication by a **constant** complex phase factor $e^{i\,\alpha}$,

$$\begin{aligned} \Phi \to \Phi' &= e^{-i\,\alpha}\Phi, \\ \Phi^\dagger \to \Phi'^\dagger &= e^{i\,\alpha}\Phi^\dagger, \end{aligned} \tag{2}$$

where $\alpha \in \mathbb{R}$. Since a complex phase is unitary i.e. the complex conjugation is also the inverse,

$$\left(e^{-i\,\alpha}\right)^\dagger = \left(e^{-i\,\alpha}\right)^{-1},$$

such phases are also called $U(1)$ factors (U stands for Unitary matrix and since a number is a 1×1 matrix, $U(1)$ is unitary matrix of size 1×1). Since this symmetry transformation does not touch spacetime but only changes the fields (configuration space variables), such a symmetry is called an **internal symmetry**. Also note that since α is a constant i.e. not a function of spacetime, it is a **global** symmetry (**global = same everywhere = independent of spacetime location**).

Check: Under the U(1) symmetry Eq. (2), the mass term is obviously invariant,

$$\begin{aligned} \Phi'^\dagger\Phi' &= \left(e^{i\,\alpha}\Phi^\dagger\right)\,\left(e^{-i\,\alpha}\Phi\right) \\ &= \Phi^\dagger\Phi \end{aligned}$$

and this is true whether α is a constant or a function of spacetime i.e. $\alpha(x)$. Now let's look at the kinetic term,

$$\begin{aligned} \left(\partial_\mu\Phi^\dagger\right)\left(\partial^\mu\Phi\right) \to \left(\partial_\mu\Phi'^\dagger\right)\left(\partial^\mu\Phi'\right) &= \partial_\mu\left(e^{i\,\alpha}\Phi^\dagger\right)\partial^\mu\left(e^{-i\,\alpha}\Phi\right), \\ &= e^{i\,\alpha}\left(\partial_\mu\Phi^\dagger\right)\,e^{-i\,\alpha}\left(\partial^\mu\Phi\right) \\ &= \left(\partial_\mu\Phi^\dagger\right)\left(\partial^\mu\Phi\right). \end{aligned}$$

So this kinetic term in the action is also invariant because α is a constant and the derivative does not act on it. If α was a function of spacetime, $\alpha = \alpha(x)$, the derivative would have acted on it and the term would not be invariant. Incidentally, a spacetime dependent phase $\alpha(x)$ is called a **local $U(1)$ transformation.**

2 Symmetries: Noether's theorem & construction of charges

Recall that Noether's theorem states that whenever a physical system has an *continuous global* symmetry i.e. when the the action functional of the system is invariant under some transformation rules of the coordinates and/or configuration space variables and the symmetry transformation parameter takes on values continuously on the real line and the parameter remains same at all points in spacetime, then there exists a conserved charge corresponding to that symmetry. In this section, we will use Noether's theorem to construct the conserved charges for the free scalar system. First we will look at spacetime symmetries such as Lorentz and translation symmetries. Since the analysis of spacetime symmetries is virtually identical for real and complex scalar field theories, we will be content to consider the *real* scalar field theory. The theory of the real scalar field, $\varphi(x)$ can be described by the action,

$$I\left[\varphi(x)\right] = \int d^4x\,\mathcal{L},$$

where the Lagrangian $\mathcal{L}$ is a function of the scalar, $\varphi(x)$ and it's spacetime derivatives $\partial_\mu\varphi(x)$,

$$\mathcal{L} = \mathcal{L}\left(\varphi(x), \partial_\mu\varphi(x)\right) = \mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi(x))$$

The integration range is over all space and time. In particular, for the **free scalar,** the Lagrangian can be taken to be,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2. \tag{3}$$

This form is dictated by Lorentz invariance i.e. the Lagrangian density **must be Lorentz scalar**. m is a Lorentz invariant quantity with the dimensions of mass (or energy). (Upon quantizing the system, the parameter, m will turn out to be the mass of the scalar field quanta/particles).

The symmetries/invariances of the real scalar field action are:

- Lorentz invariance $x \to x' = \Lambda x$, $\varphi(x) \to \varphi'(x') = \varphi(x)$.
- Translation invariance $x \to x' = x + a$, $\varphi(x) \to \varphi'(x') = \varphi(x)$.
- Discrete internal symmetry, $\varphi \to \varphi' = -\varphi$. (Only if the lagrangian contains **even powers** of φ).

Checks:

- Lorentz invariance is rather obvious because most terms in the action is Lorentz invariant, d^4x, m^2, φ^2. Even the kinetic term, $\partial_\mu\varphi\,\partial^\mu\varphi$, is because the Lorentz index μ is contracted, viz:

$$\partial_\mu\varphi(x) \to \partial'_\mu\varphi'(x') = \Lambda_\mu{}^{\nu}\,\partial_\nu\varphi(x),$$

$$\begin{aligned}
\partial_\mu\varphi(x)\,\partial^\mu\varphi(x) \to \partial'_\mu\varphi'(x')\,\partial'^\mu\varphi'(x') &= \Lambda_\mu{}^{\nu}\,\partial_\nu\varphi(x)\,\Lambda^\mu{}_\alpha\partial^\alpha\varphi(x) \\
&= (\Lambda_\mu{}^{\nu})\,(\Lambda^\mu{}_\alpha)\,\partial_\nu\varphi(x)\,\partial^\alpha\varphi(x) \\
&= \delta^\nu_\alpha\,\partial_\nu\varphi(x)\,\partial^\alpha\varphi(x) \\
&= \partial_\nu\varphi(x)\,\partial^\nu\varphi(x).
\end{aligned}$$

  Thus the action

$$\begin{aligned}
I\left[\varphi'(x')\right] &= \int d^4x' \left[\frac{1}{2}\partial'_\mu\varphi'(x')\partial'^\mu\varphi'(x') - \frac{1}{2}m^2\varphi^2(x)\right] \\
&= \int d^4x \left[\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \frac{1}{2}m^2\varphi^2(x)\right] \\
&= I\left[\varphi(x)\right],
\end{aligned}$$

  remains invariant.

- Translation invariance is also obvious because the action integral being defined over all space and time i.e ranges of integration being $(-\infty, \infty)$, is independent of the origin of coordinates and there is no *explicit* dependence on the coordinates, x. Recall that under translations, namely,

$$x \to x' = x + a,$$

  the field φ transforms as,

$$\varphi'(x') = \varphi(x),$$

  What about the kinetic piece containing terms such as $\partial_\mu\varphi$. Such a term seems to care about the spacetime coordinate through the derivative, $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Actually even this derivative is independent of the shift in origin because under a shift of origin of coordinates,

$$x \to x' = x + a,$$

  Conversely,

$$x = x' - a$$

the derivative transforms as

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}, \\
= \frac{\partial\left(x'^\nu - a^\nu\right)}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \\
= \frac{\partial x'^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \\
= \partial^\nu_\mu \frac{\partial}{\partial x^\nu} \\
= \frac{\partial}{\partial x^\mu}.$$

So the derivative remains unchanged. (Here since a is a constant its derivative vanishes, and we get $\frac{\partial\left(x'^\nu - a^\nu\right)}{\partial x'^\mu} = \frac{\partial x'^\nu}{\partial x'^\mu}$).

- Discrete internal symmetry such as $\varphi \to \varphi' = -\varphi$ is also obvious when the Lagrangian contains even powers of φ. However as they are discrete (non-continuous) and do not obey Noether's theorem and they will not give rise to conserved charges.

In the case of spacetime symmetries such as Lorentz transformations and translations, we see that the parameters $\Lambda^\mu{}_\nu$ and a^μ are indeed constants and not functions of spacetime i.e. these are **global** symmetries. So Noether's theorem applies in these cases and tells us that there must be conserved charges. Here we obtain the expressions for those charges using the so called **"*Noether procedure*"**. In a nutshell, the algorithm for extracting Noether charges is as follows:

1. First make the global symmetry parameter, say ε, infinitesimal. This is allowed because the symmetry parameter takes values on some continuous segment of the real line which *includes the origin*. In case of translations we will take the shift parameter, a^μ to be infinitesimal, and in case of rotations the angle of rotation, θ is to be taken infinitesimal. When this done, in all subsequent steps of the procedure we will only keep terms which are up to $O(\varepsilon)$, i.e. linear order in the infinitesimal symmetry parameter. Higher order terms i.e. $O\left(\varepsilon^2\right)$ will be dropped.

2. Next we will ***temporarily assume*** that the symmetry parameter ε, is a function of spacetime, i.e. $\varepsilon = \varepsilon(x)$. In case of Lorentz transformations, we will temporarily make $\Lambda^\mu{}_\nu = \Lambda^\mu{}_\nu(x) = \delta^\mu_\nu + \omega^\mu{}_\nu(x)$ and in the case of translations $a^\mu = a^\mu(x)$. Then (**to lowest order**), the change in the action integral should be,
$$\delta I = -\int d^4x\ \left(\partial_\mu \varepsilon(x)\right)\ j^\mu + O\left(\varepsilon^2\right).$$
Again for example for translations one must have the change in action,
$$\delta I = -\int d^4x\ \left(\partial_\mu a^\nu(x)\right)\ \theta^\mu{}_\nu + O\left(a^2\right),$$
while for Lorenz transformations we must have,
$$\delta I = -\int d^4x\ \left(\partial_\rho \omega^\mu{}_\nu(x)\right)\ M^\mu{}_{\nu\rho} + O\left(\omega^2\right).$$
This form is consistent with the our expectation for a global symmetry. Right now, the symmetry parameter is not global since we have temporarily assumed it (them) to be functions of spacetime, and this is ***NOT*** a symmetry of the action, the hence action must have a non-vanishing change. However, in the special case when a and ω are constants these expressions must vanish as the action is supposed to invariant under the global/constant changes which is a symmetry of the system.

3. From the expression for the δI, read off the companion coefficient terms i.e. $T^{\mu}{}_{\nu}$ or $M^{\mu}{}_{\nu\rho}$ which are some functions of the field and it's derivatives. These are the conserved Noether currents! They will obey a continuity type equation when the equations of motion hold. This can be inferred from the above expressions for δI by a simple integration by parts and abandoning the total derivative term (we can abandon this term under the assumption that the surface term goes to zero at infinity). For example, for the translation invariance:

$$\begin{aligned}\delta I &= -\int d^4x\ (\partial_\mu a^\nu(x))\ \theta^{\mu}{}_{\nu} \\ &= -\int d^4x\, \partial_\mu\left(a^\nu(x)\,\theta^{\mu}{}_{\nu}\right) + \int d^4x\ (\partial_\mu \theta^{\mu}{}_{\nu})\ a^\nu(x) \\ &= \int d^4x\ (\partial_\mu \theta^{\mu}{}_{\nu})\ a^\nu(x).\end{aligned}$$

where the total derivative gives rise to a surface term at infinity which is assumed to vanish[1]. More generally (not just for translations),

$$\delta I = -\int d^4x\ (\partial_\mu \varepsilon(x))\ j^\mu = \int d^4x\ \varepsilon(x)\,\partial_\mu j^\mu.$$

4. Use the variational principle to demand the change in the action to vanish around classical configurations (whereby the equation of motion holds). The change of the action might not vanish when the symmetry parameter is turned local (function of spacetime) but now since we are talking about configurations around the equation of motion, the variation of the action has to vanish for *arbitrary* variations, *including* the case when the variation happens to be with the symmetry parameter being local. Thus when equations of motion hold, one has

$$\delta I = 0,$$

or,

$$\int d^4x\ \varepsilon(x)\,\partial_\mu j^\mu = 0$$

The only way this integral can vanish for arbitrary function $\varepsilon(x)$ is when rest of the integrand vanishes, i.e.

$$\partial_\mu j^\mu = 0.$$

This is nothing but the continuity equation for a current density, j^μ! We know from past experience that it represents a conservation law. More concretely say for translations, one has when the equation of motion holds,. a^ν becomes a constant, it can be pulled out of the integral and we have the expression,

$$\delta I = a^\nu \int d^4x\ (\partial_\mu \theta^{\mu}{}_{\nu})\ a^\nu(x) = 0.$$

which immediately leads to,

$$\partial_\mu \theta^{\mu}{}_{\nu} = 0,$$

the continuity equation for a current density. This current density corresponding to translation/shift symmetry of a field theory is called the *Stress-Energy-Momentum* tensor and is denoted by $\theta^{\mu}{}_{\nu}$ or raising both indices, $\theta^{\mu\nu}$. One can explicitly check, using the equations of motion of the fields, that the above continuity equation indeed holds.

[1] If this boundary term does not vanish even with the use of boundary conditions, i.e. $\int d^4x\ \partial_\mu\left(a^\nu(x)\,\theta^{\mu}{}_{\nu}\right) = \int_{S^\infty} d^\mu S\ \left(a^\nu(x)\,\theta^{\mu}{}_{\nu}\right) \not\to 0$, they will modify the definition of the Noether charge. This is actually not very rare, in fact in general relativity this is a commonplace scenario.

5. Construct the Noether charge by performing the spatial volume integral

$$Q = \int d^3\mathbf{x}\, j^0.$$

where j^μ is a conserved Noether current-density. This follows easily from the continuity equation.

$$\partial_\mu j^\mu = 0 \Rightarrow \frac{\partial j^0}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0.$$

Which implies,

$$\frac{\partial j^0}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{j}.$$

Taking a (spatial) volume integral of both sides,

$$\int d^3\mathbf{x}\, \frac{\partial j^0}{\partial t} = -\int d^3\mathbf{x}\, \left(\boldsymbol{\nabla} \cdot \mathbf{j}\right).$$

Now in the LHS we swap the space-integral and time derivative, $\int d^3\mathbf{x}\, \frac{\partial j^0}{\partial t} = \frac{d}{dt}\left(\int d^3\mathbf{x}\, j^0\right)$, one the RHS we convert it to a surface integral at spatial infinity using Gauss divergence theorem,

$$\frac{d}{dt}\left(\int d^3\mathbf{x}\, j^0\right) = -\int_{S^\infty} dS\, \hat{\mathbf{n}} \cdot \mathbf{j}.$$

Using the right boundary conditions the surface term at spatial infinity on the RHS vanishes and we leads to the conservation law,

$$\frac{d}{dt}\left(\underbrace{\int d^3\mathbf{x}\, j^0}_{=Q}\right) = 0.$$

For example, for the case of translations, the conserved Noether charge is,

$$P_\nu = \int d^3\mathbf{x}\, \theta^0{}_\nu,$$

which are nothing but the linear momentum 4-vector.

2.1 The "Stress-Energy-Momentum" tensor for the (real) scalar field theory

We use the Noether procedure to extract the stress-energy-momentum tensor for the (real) scalar field theory. First step is to turn the shift parameter infinitesimal. Then we set the infinitesimal symmetry parameter to not be constant (global), but instead a function of spacetime,

$$a^\mu = a^\mu(x).$$

So we have new coordinates,

$$x \to x'^\mu = x^\mu + a^\mu(x).$$

The Jacobian matrix components for the change of variables, $x \to x'$ are,

$$J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu a^\mu,$$

i.e. it is a sum of the identity matrix and a small change, $\partial_\nu a^\mu$. Hence to first order of the shift, the Jacobian determinant is,

$$\begin{aligned} |J| &= 1 + \mathrm{trace}(\partial_\nu a^\mu) \\ &= 1 + \partial_\rho a^\rho. \end{aligned} \tag{4}$$

The derivatives transform like,

$$\begin{aligned}\partial_\mu \to \partial'_\mu &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu \\ &= \left(\delta^\nu_\mu - \partial_\mu a^\nu\right)\,\partial_\nu \\ &= \partial_\mu - \partial_\mu a^\nu\,\partial_\nu. \end{aligned} \tag{5}$$

Now let's look at the change in φ and it's derivatives, $\partial_\mu\varphi$. We have,

$$\varphi'(x') = \varphi(x),$$

while,

$$\begin{aligned}\partial'_\mu\varphi'(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu\varphi(x) \\ &= \partial_\mu\varphi(x) - \partial_\mu a^\nu\,\partial_\nu\varphi(x). \end{aligned} \tag{6}$$

Now we are ready to compute the transformed action after this spacetime dependent translation, $a(x)$ using the transformation equations (4-6) :

$$\begin{aligned} I\left[\varphi'(x')\right] &= \int d^4x'\,\mathcal{L}\left(\varphi'(x'),\partial'_\mu\varphi'(x')\right) \\ &= \int d^4x\,|J|\,\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x) - \partial_\mu a^\nu\,\partial_\nu\varphi(x)\right) \\ &= \int d^4x\,\left(1+\partial_\rho a^\rho\right)\left[\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) - \partial_\mu a^\nu\,\partial_\nu\varphi(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi(x)\right)} + O(a^2)\right] \\ &= \int d^4x\,\left[\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) + \partial_\rho a^\rho\,\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) - \partial_\mu a^\nu\,\partial_\nu\varphi(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi(x)\right)} + O(a^2)\right] \\ &= I\left[\varphi(x)\right] + \int d^4x\,\left[\partial_\rho a^\rho\,\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) - \partial_\mu a^\nu\,\partial_\nu\varphi(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi(x)\right)}\right] + O(a^2), \end{aligned}$$

This implies, to first order in a

$$\begin{aligned} \delta I \equiv I\left[\varphi'(x')\right] - I\left[\varphi(x)\right] &= \int d^4x\,\left[\partial_\rho a^\rho\,\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) - \partial_\mu a^\nu\,\partial_\nu\varphi(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi(x)\right)}\right] \\ &= \int d^4x\,\partial_\mu a^\nu\left[\delta^\mu_\nu\mathcal{L}\left(\varphi(x),\partial_\mu\varphi(x)\right) - \partial_\nu\varphi(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi(x)\right)}\right] \\ &= -\int d^4x\,\partial_\mu a^\nu\,\theta^\mu{}_\nu, \end{aligned}$$

where we have identified the conserved current,

$$\theta^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi(x)\right.}\partial_\nu\varphi(x) - \delta^\mu_\nu\mathcal{L}.$$

Raising the ν-index we arrive at the expression of the *canonical* stress-energy-momentum tensor,

$$\begin{aligned} \theta^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi(x)\right.}\partial^\nu\varphi(x) - \eta^{\mu\nu}\mathcal{L}, \\ &= \left(\partial^\mu\varphi\right)\left(\partial^\nu\varphi\right) - \eta^{\mu\nu}\mathcal{L}. \end{aligned} \tag{7}$$

We note couple of things about this canonical stress tensor,

- One can go ahead and check, using the equations of motion, that this is indeed conserved that it satisfies the continuity equation, $\partial_\mu \theta^{\mu\nu} = 0$,

$$\begin{aligned}\partial_\mu \theta^{\mu\nu} &= \partial_\mu \left(\partial^\mu \varphi\right)\left(\partial^\nu \varphi\right) - \eta^{\mu\nu}\partial_\mu \mathcal{L} \\ &= \left(\partial^2 \varphi\right)\left(\partial^\nu \varphi\right) + \left(\partial^\mu \varphi\right)\left(\partial_\mu \partial^\nu \varphi\right) - \partial^\nu \mathcal{L}\end{aligned}$$

  We simplify the first term using the equations of motion,

$$\begin{aligned}\partial^2 \phi &= -\frac{\partial V}{\partial \varphi}, \\ \Rightarrow \left(\partial^2 \varphi\right)\left(\partial^\nu \varphi\right) &= -\frac{\partial V}{\partial \varphi}\partial^\nu \varphi \\ &= -\partial^\nu V\end{aligned}$$

  while the second term can be rewritten as

$$\left(\partial^\mu \varphi\right)\left(\partial_\mu \partial^\nu \varphi\right) = \left(\partial^\mu \varphi\right)\partial^\nu\left(\partial_\mu \varphi\right) = \partial^\nu\left(\frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi\right)$$

  Thus, the first two terms add up to,

$$\begin{aligned}\left(\partial^2 \varphi\right)\left(\partial^\nu \varphi\right) + \left(\partial^\mu \varphi\right)\left(\partial_\mu \partial^\nu \varphi\right) &= -\partial^\nu V + \partial^\nu\left(\frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi\right) \\ &= \partial^\nu\left(\frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - V\right) \\ &= \partial^\nu \mathcal{L},\end{aligned}$$

  which cancels the third term, giving us,

$$\partial_\mu \theta^{\mu\nu} = 0.$$

- The conserved charges corresponding to the current are nothing but the components of the four-momentum, P^μ, i.e.

$$P^\nu = \int d^3\mathbf{x}\, \theta^{0\nu}.$$

  Let's evaluate the 00-component i.e. *energy density*, θ^{00},

$$\begin{aligned}\theta^{00} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \varphi(x)\right)}\partial^0 \varphi(x) - \eta^{00}\mathcal{L} \\ &= \left(\partial^0 \varphi\right)^2 - \mathcal{L}, \\ &= \left(\partial^0 \varphi(x)\right)^2 - \frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi + V(\varphi), \\ &= \left(\partial_0 \varphi\right)^2 - \frac{1}{2}\left(\partial_0 \varphi\right)^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2 + V(\varphi), \\ &= \frac{1}{2}\left(\partial_0 \varphi\right)^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\varphi\right)^2 + V(\varphi).\end{aligned}$$

  This expression being a sum of squares is manifestly positive. This is reassuring because we would want a free system to have energy positive semi-definite.

- This tensor is symmetric between the indices, μ, ν. This is only true for scalar fields. For the Maxwell field, we will see that the corresponding stress tensor will *not* be symmetric.

- Note that the stress-energy-momentum is non-unique to some extent. One can always add a term like, $\partial_\lambda B^{\lambda\mu\nu}$ where B is a tensor that has the following antisymmetric properties,

$$B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}.$$

The new quantity[2],

$$T^{\mu\nu} = \theta^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu}$$

is also conserved,

$$\partial_\mu T^{\mu\nu} = \partial_\mu \theta^{\mu\nu} + \partial_\mu \partial_\lambda B^{\lambda\mu\nu} = 0.$$

For the Maxwell field, one can exploit this ambiguity to define a stress tensor which is symmetric in the indices, μ and ν,

$$T^{\mu\nu} = T^{\nu\mu}.$$

Before we do that we need to first obtain the expression for the charges conserved as a result of Lorentz invariance.

Homework: Follow the Noether procedure to construct the conserved charges for the scalar field theory for symmetry under Lorentz transformations,

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu \, x^\nu.$$

[2]For those who are interested there is a special "symmetrizing improvement term", $B^{\lambda\mu\nu}$ is called the Belinfante-Rosenfield term after the two people who independently arrived at the expression. For the Maxwell field we will see that the variation of action under Lorentz transformation automatically gives us the improved symmetric stress tensor.