Lecture 2

The Steady Magnetic Field

Electromagnetic Field Theory

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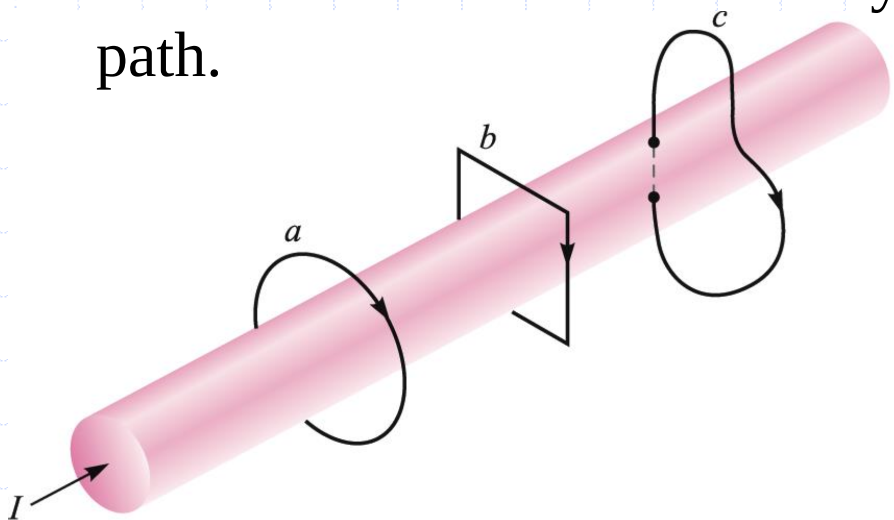
Ampere's Circuital Law

In **electrostatics problems** that featured a lot of symmetry we were able to apply **Gauss's Law** to solve for the electric field intensity much more easily than applying **Coulomb's Law**.

Likewise, in **magnetostatic** problems with sufficient symmetry we can employ **Ampere's Circuital Law** more easily than **the Law of Biot-Savart**.

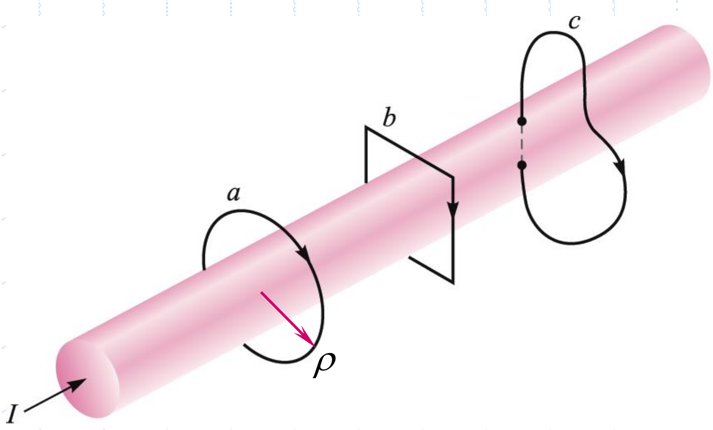
Ampere's Circuital Law states that the line integral of **H** about *any closed path* is exactly equal to the **direct current** enclosed by that path.

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$



Ampere's Law Applied to a Long Wire

In the figure, the integral of **H** about closed paths *a* and *b* gives the total current *I*, while the integral over path *c* gives only that portion of the current that lies within *c*.



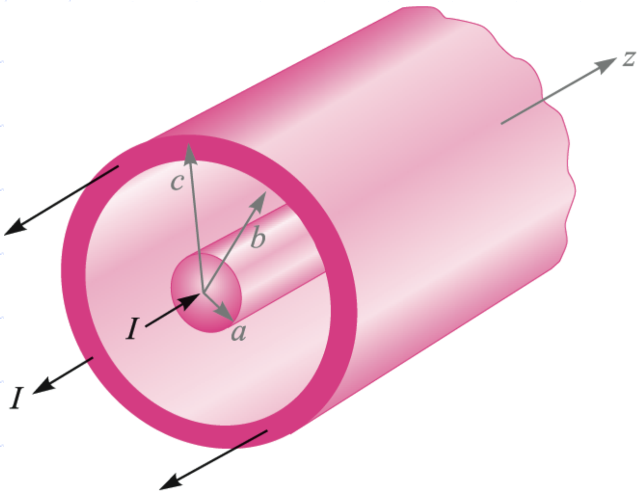
Symmetry suggests that **H** will be circular, constant-valued at constant radius, and centered on the current (*z*) axis.

Choosing path *a*, and integrating **H** around the circle of radius ρ gives the enclosed current, *I*:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

so that: $$H_\phi = \frac{I}{2\pi\rho}$$ as before.

Coaxial Transmission Line



In the coax line, we have two concentric *solid* conductors that carry **equal** and **opposite** currents, I.

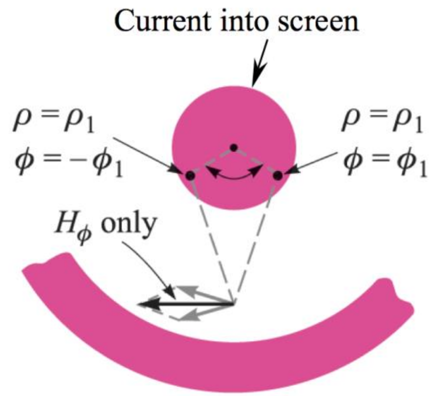
The line is assumed to be **infinitely long**, and the circular symmetry suggests that **H** will be **entirely ϕ - directed**, and will vary only with radius ρ.

Our objective is to find the magnetic field for all values of ρ

Field Between Conductors

The inner conductor can be thought of as made up of a **bundle of filament currents**, each of which produces the field of a long wire.

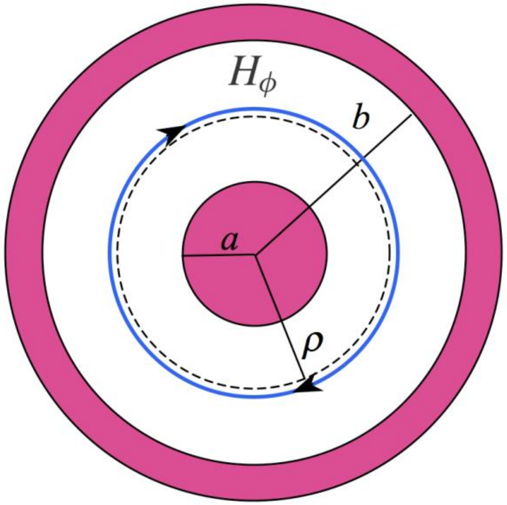
Consider **two** such filaments, located at the same radius from the *z* axis, ρ_1, but which lie at symmetric ϕ coordinates, ϕ_1 and $-\phi_1$. Their field contributions **superpose** to give a **net** H_ϕ component as shown.



The same happens for every pair of symmetrically-located filaments, which taken as a whole, make up the entire center conductor.

The field between conductors is thus found to be the same as that of filament conductor on the *z* axis that carries current, *I*. Specifically:

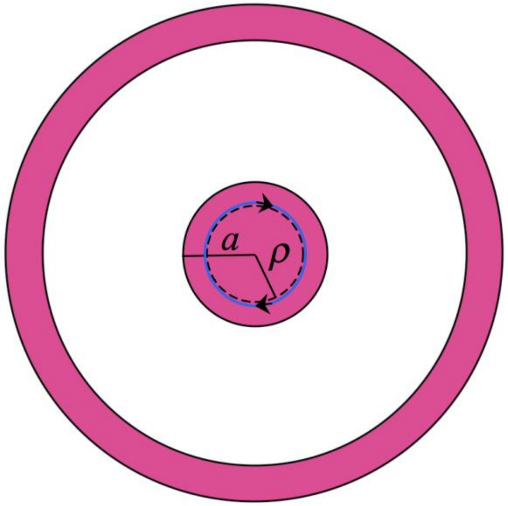
$$H_\phi = \frac{I}{2\pi\rho} \qquad a < \rho < b$$



Field Within the Inner Conductor

With current uniformly distributed inside the conductors, the **H** can be assumed circular everywhere.

Inside the inner conductor, and at radius ρ, we again have:



$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi 2\pi \rho$$

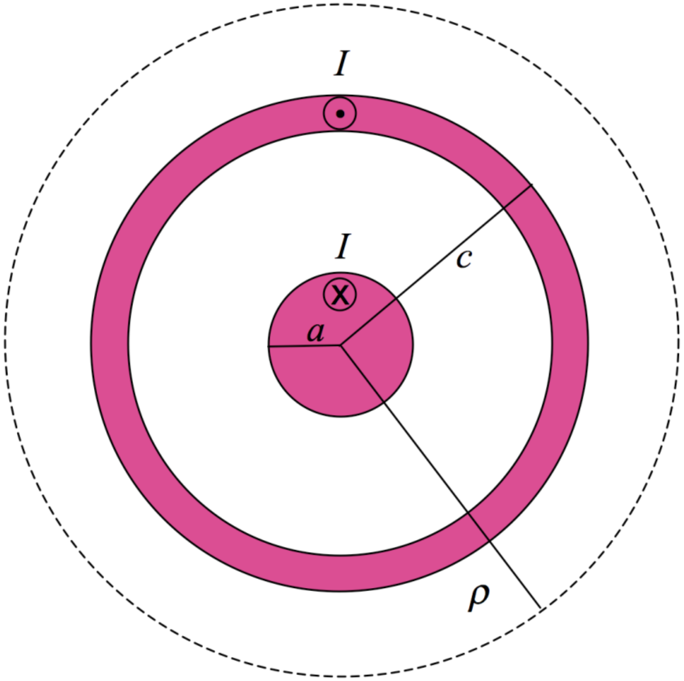
But now, the current enclosed is $I_{\text{encl}} = I \dfrac{\rho^2}{a^2}$

so that $2\pi \rho H_\phi = I \dfrac{\rho^2}{a^2}$ or finally: $H_\phi = \dfrac{I\rho}{2\pi a^2} \quad (\rho < a)$

Field Outside Both Conducors

Outside the transmission line, where $\rho > c$, no current is enclosed by the integration path, and so

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0$$



As the current is uniformly distributed, and since we have circular symmetry, the field would have to be constant over the circular integration path, and so it must be true that:

$$H_\phi = 0 \quad (\rho > c)$$

Field Inside the Outer Conductor

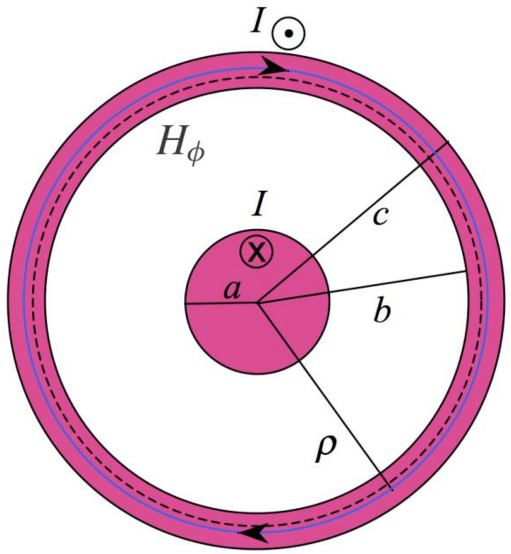
Inside the outer conductor, the enclosed current consists of that within the inner conductor plus that portion of the outer conductor current existing at radii less than ρ

Ampere's Circuital Law becomes

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

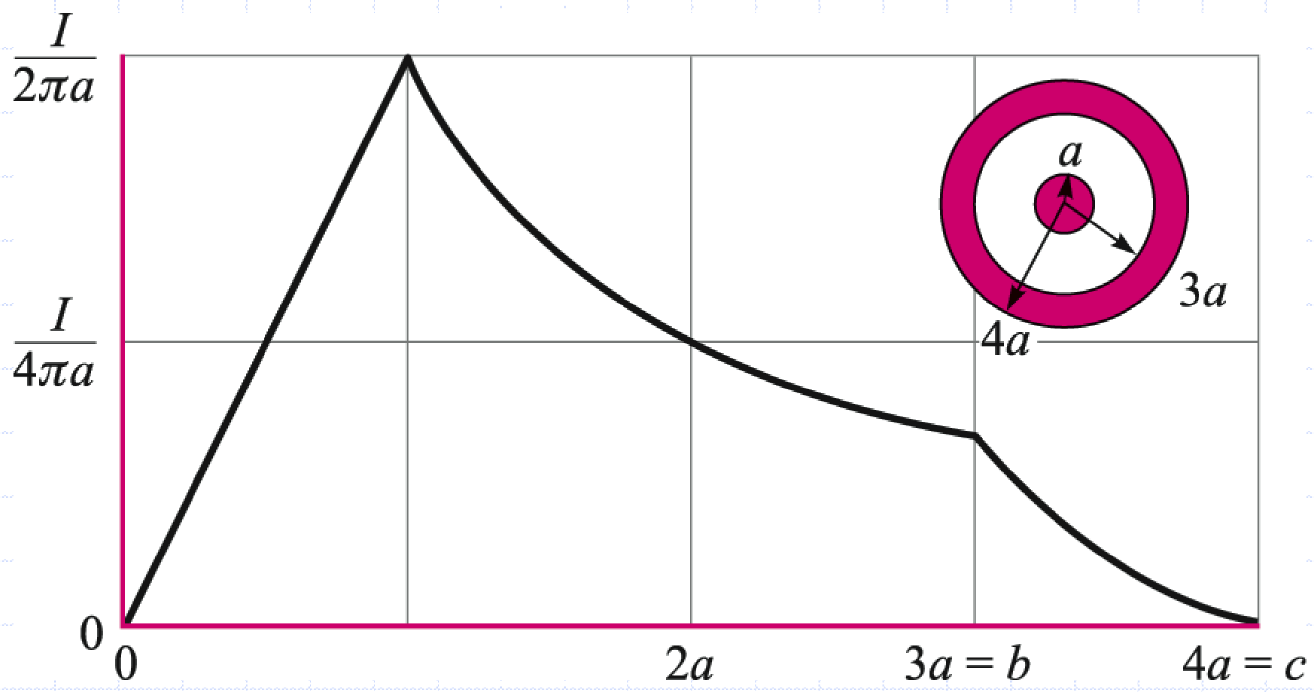
..and so finally:

$$H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \qquad (b < \rho < c)$$



Magnetic Field Strength as a Function of Radius in the Coax Line

Combining the previous results, and assigning dimensions as shown in the inset below, we find:



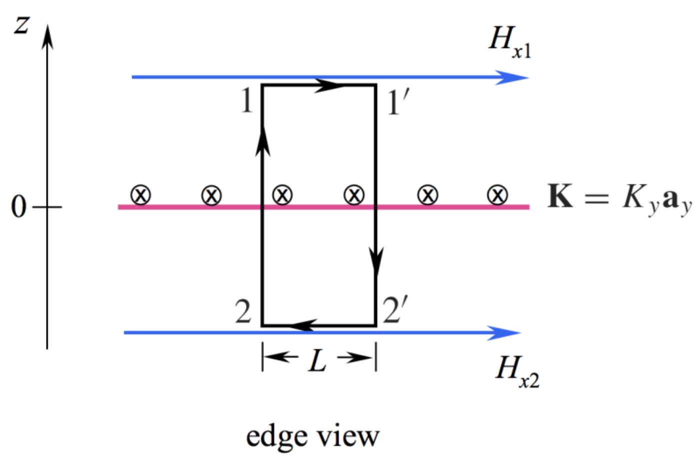
Magnetic Field Arising from a Current Sheet

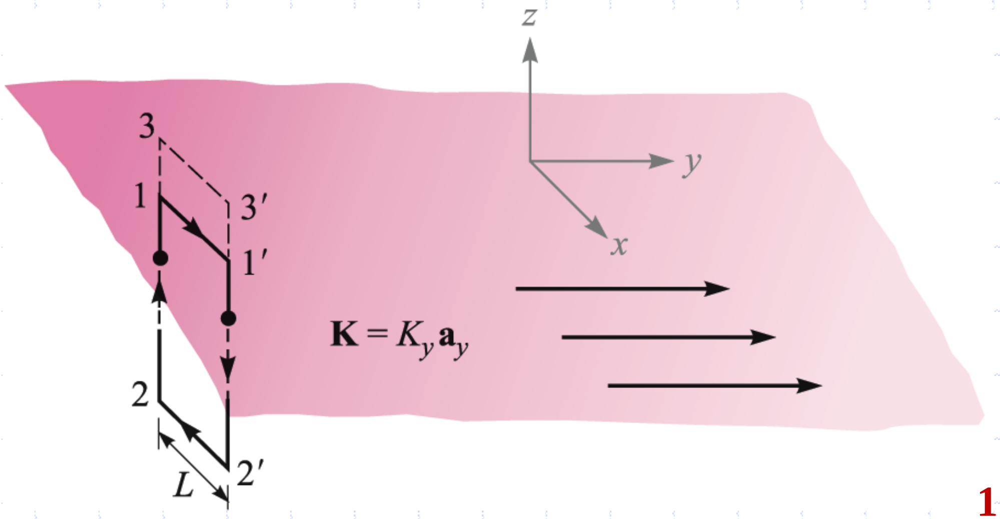
For a uniform plane current in the y direction, we expect an x-directed **H** field from symmetry.

Applying **Ampere's circuital law** to the path $1-1'-2'-2-1$, we find:

$$H_{x1}L + H_{x2}(-L) = K_y L \quad \text{or} \quad H_{x1} - H_{x2} = K_y$$

In other words, the magnetic field is discontinuous across the current sheet by the magnitude of the surface current density.



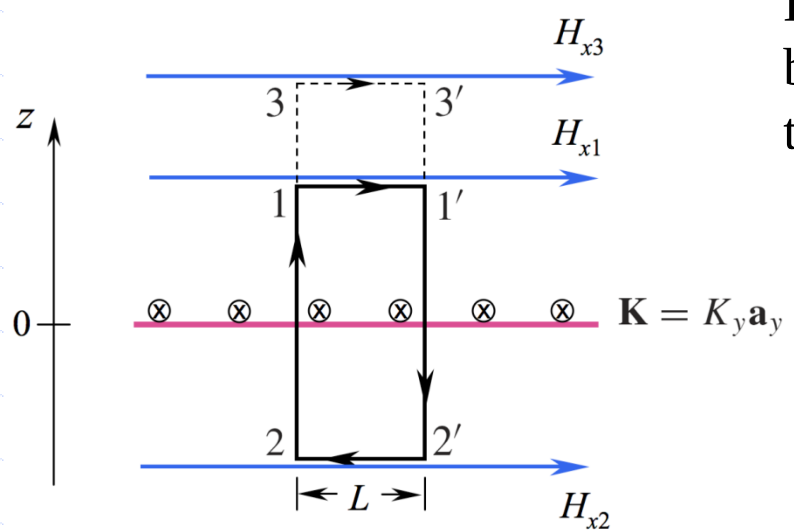


Magnetic Field Arising from a Current Sheet

If instead, the upper path is elevated to the line between 3 and 3′, the same current is enclosed and we would have

$H_{x3} - H_{x2} = K_y$ from which we conclude that $H_{x3} = H_{x1}$

so the field is *constant in each region* (above and below the current plane)



edge view

By symmetry, the field above the sheet must be the same in magnitude as the field below the sheet. Therefore, we may state that

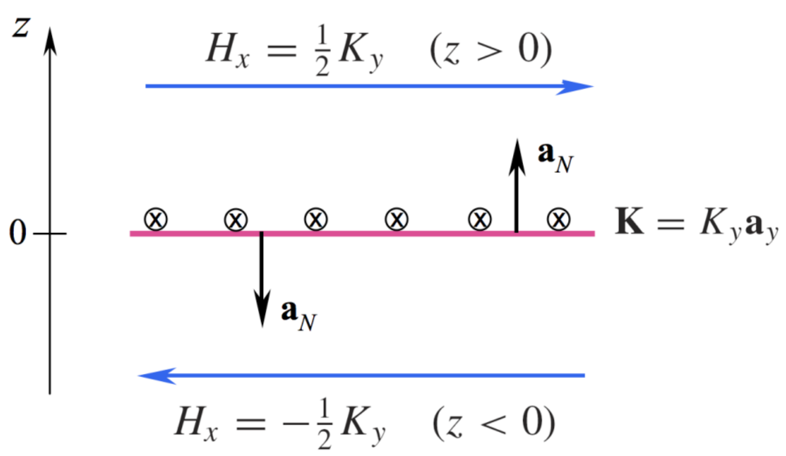
$$H_x = \frac{1}{2} K_y \quad (z > 0)$$

and $$H_x = -\frac{1}{2} K_y \quad (z < 0)$$

Magnetic Field Arising from a Current Sheet

The actual field configuration is shown below, in which magnetic field above the current sheet is equal in magnitude, but in the direction opposite to the field below the sheet.

The field in either region is found by the cross product:

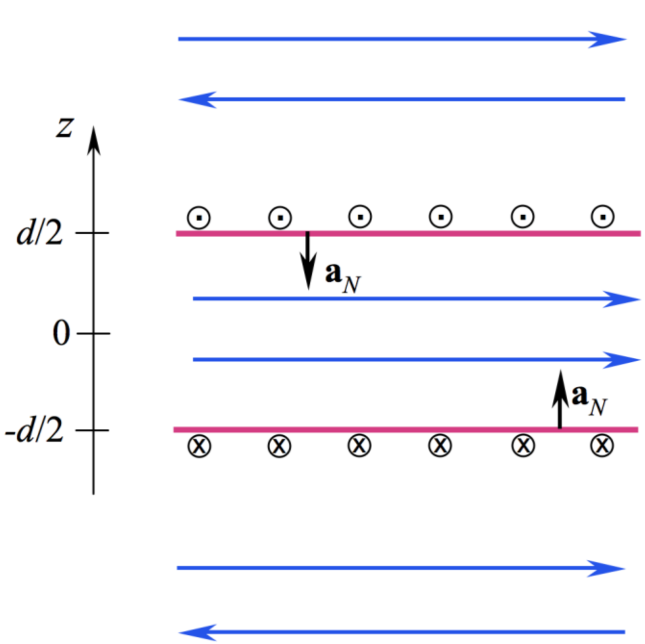


$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N$$

where $\mathbf{a}_N$ is the unit vector that is normal to the current sheet, and that points into the region in which the magnetic field is to be evaluated.

Magnetic Field Arising from Two Current Sheets

Here are two parallel currents, **equal and opposite**, as you would find in a **parallel-plate transmission line**. If the sheets are much wider than their spacing, then the magnetic field will be contained in the region between plates, and will be nearly zero outside.



$H_{x2}\,(z > d/2)$

$H_{x1}\,(z > d/2)$

These fields cancel for current sheets of infinite width.

$\mathbf{K}_1 = -K_y\,\mathbf{a}_y$

$H_{x1}\,(-d/2 < z < d/2)$

$H_{x2}\,(-d/2 < z < d/2)$

These fields are equal and add to give

$$\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (-d/2 < z < d/2)$$

where **K** is either $\mathbf{K}_1$ or $\mathbf{K}_2$

$\mathbf{K}_2 = -K_y\,\mathbf{a}_y$

$H_{x1}\,(z < -d/2)$

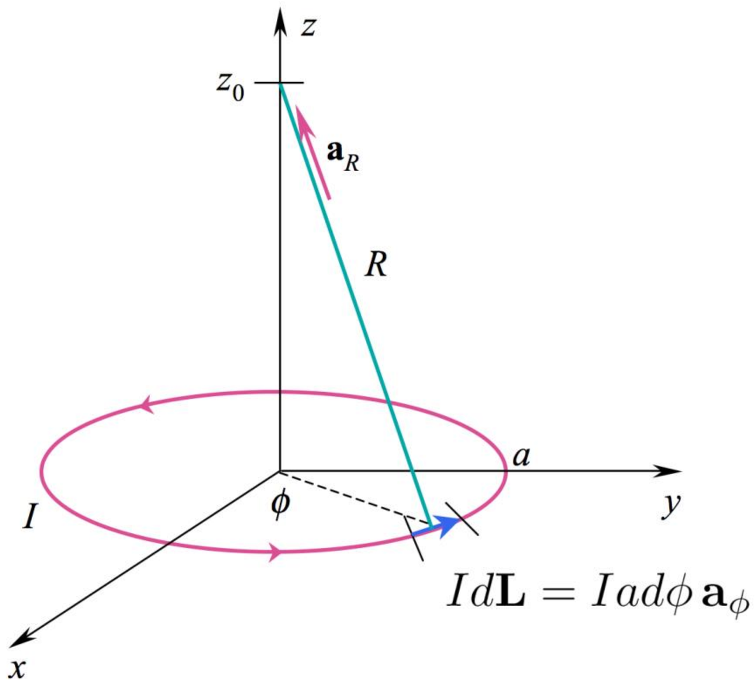
$H_{x2}\,(z < -d/2)$

These fields cancel for current sheets of infinite width.

Current Loop Field

Using the **Biot-Savart Law**, we previously found the magnetic field on the *z* axis from a circular current loop:

$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$



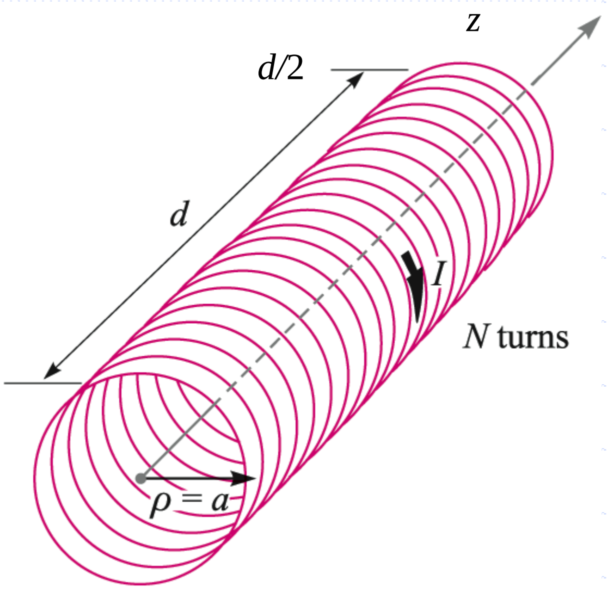
We will now use this result as a **building block** to construct the magnetic field on the axis of a solenoid -- formed by a stack of identical current loops, centered on the *z* axis.

On-Axis Field Within a Solenoid

We consider the single current loop field as a differential contribution to the total field from a stack of *N* closely-spaced loops, each of which carries current *I*. The length of the stack (solenoid) is *d*, so therefore the ***density of turns*** will be *N/d*.

Now the current in the turns within a differential length, *dz*, will be

$$dI = \frac{N}{d} I dz$$



We consider this as our **differential "loop current"**

so that the previous result for **H** from a single loop:

$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$

now becomes:

$$d\mathbf{H} = \frac{(N/d) I dz (\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z^2)^{3/2}}$$

in which *z* is measured from the center of the coil, where we wish to evaluate the field.

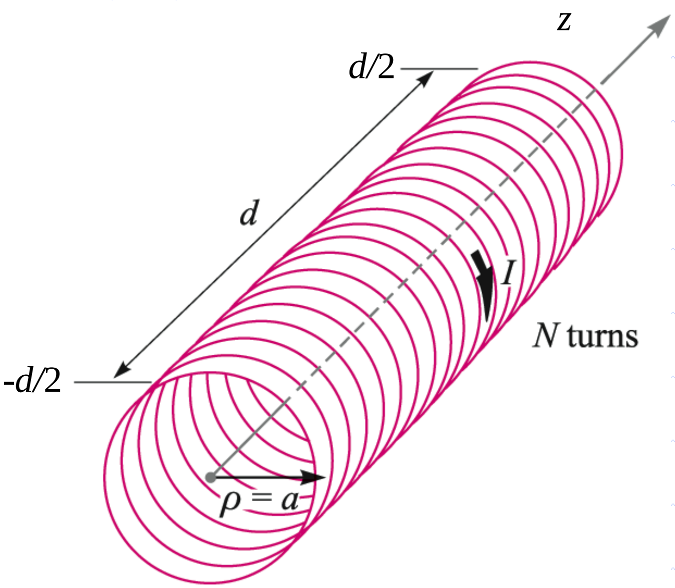
Solenoid Field, Continued

The **total field on the *z* axis at *z* = 0** will be the sum of the field contributions from all turns in the coil -- or the integral of $d\mathbf{H}$ over the length of the solenoid.

$$\mathbf{H} = \int d\mathbf{H} = \int_{-d/2}^{d/2} \frac{(N/d) I dz (\pi a^2)\mathbf{a}_z}{2\pi (a^2 + z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\mathbf{a}_z \int_{-d/2}^{d/2} \frac{dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\mathbf{a}_z \frac{d}{a^2\sqrt{a^2+(d/2)^2}} = \frac{NI\,\mathbf{a}_z}{2\sqrt{a^2+(d/2)^2}}$$

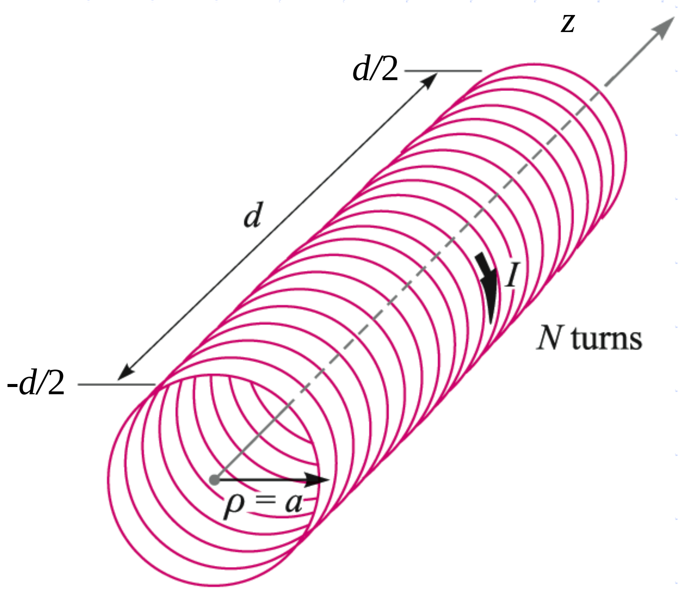


$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$$

Approximation for Long Solenoids

We now have the on-axis field at the solenoid midpoint ($z = 0$):

$$\mathbf{H} = \frac{NI\,\mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}}$$



Note that for **long solenoids**, for which $d \gg a$, the result simplifies to:

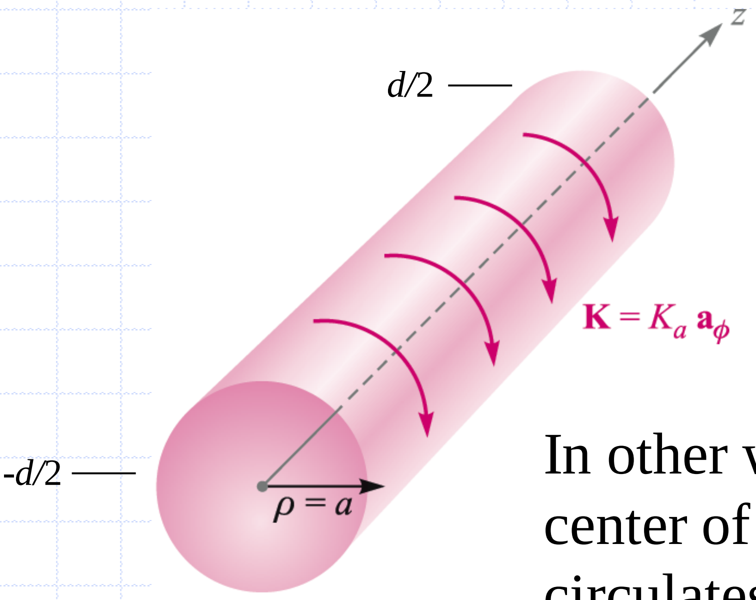
$$\mathbf{H} \doteq \frac{NI}{d}\mathbf{a}_z \quad (d >> a)$$

This result is valid at all **on-axis positions** deep within long coils -- at distances from each end of several radii.

Another Interpretation: Continuous Surface Current

The solenoid of our previous example was assumed to have many tightly-wound turns, with several existing within a differential length, *dz*. We could model such a current configuration as a continuous surface current of density $\mathbf{K} = K_a \mathbf{a}_\phi$ A/m.

$$\mathbf{K} = K_a \mathbf{a}_\phi = \frac{NI}{d} \mathbf{a}_\phi \quad \text{A/m}$$



Therefore:

$$\mathbf{H}(\rho = z = 0) = \frac{K_a d\, \mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}}$$

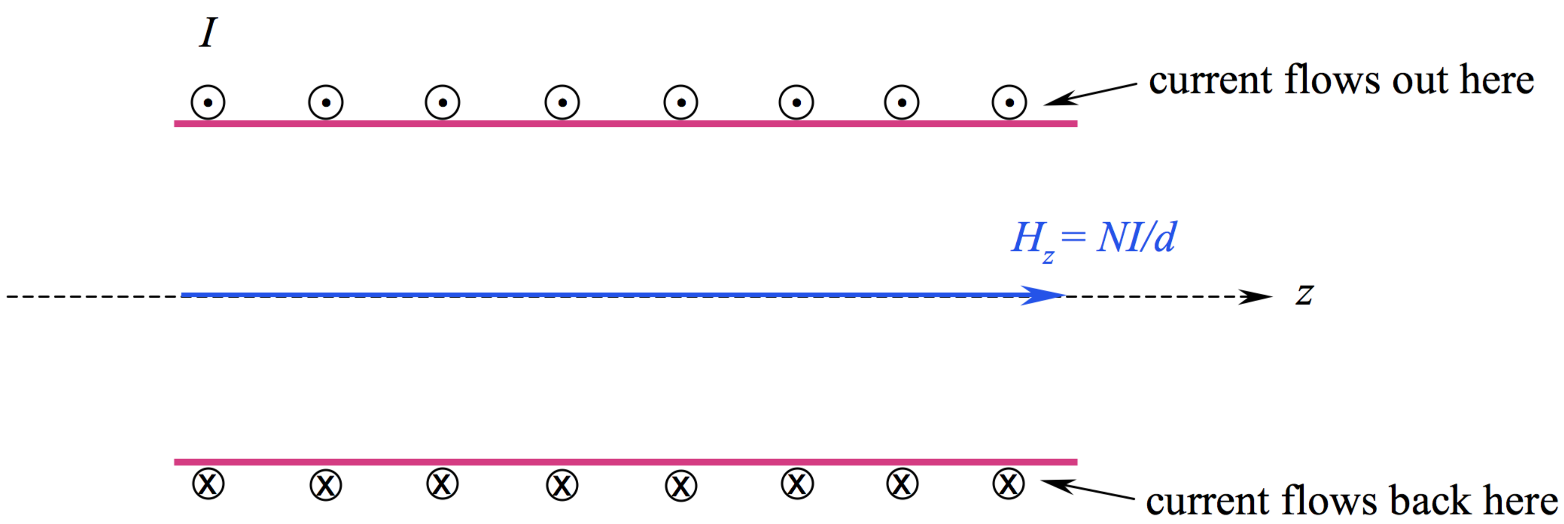
$$\doteq K_a \mathbf{a}_z \ (d >> a) \quad \text{A/m}$$

In other words, the <u>on-axis field magnitude</u> near the center of a cylindrical current sheet, where current circulates around the *z* axis, and whose length is much greater than its radius, is <u>just the surface current density</u>.

Solenoid Field -- Off-Axis

To find the field within a solenoid, but **off the *z* axis**, we apply Ampere's Circuital Law in the following way:

The illustration below shows the **solenoid cross-section**, from a lengthwise cut through the *z* axis. Current in the windings flows in and out of the screen in the circular current path.
Each turn carries current *I*. The magnetic field along the *z* axis is *NI/d* as we found earlier.

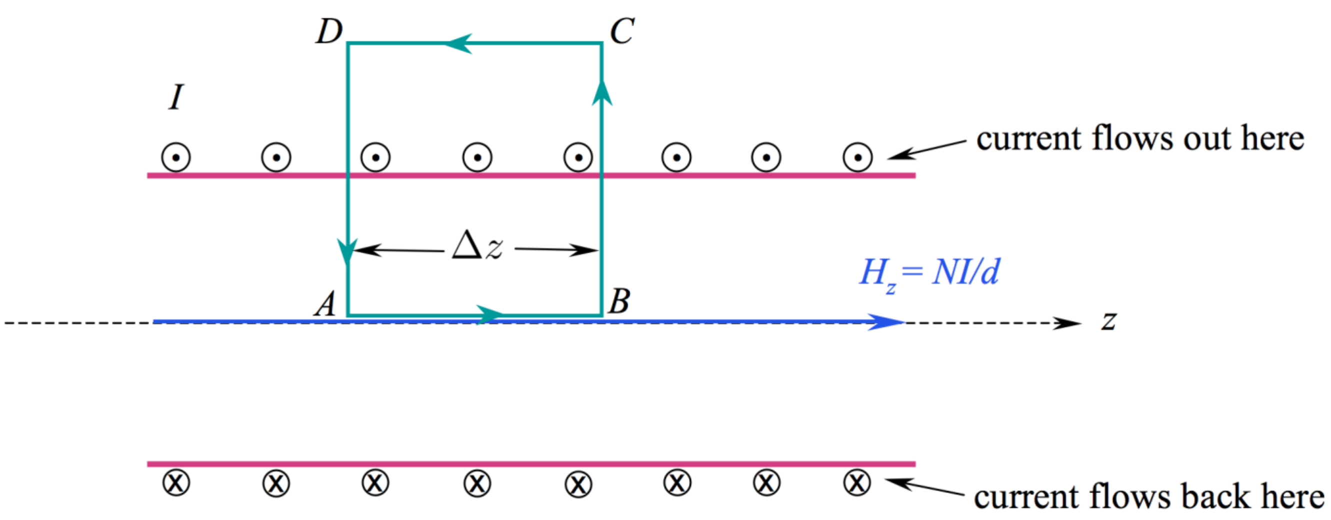


Application of Ampere's Law

Applying Ampere's Law to the rectangular path shown below leads to the following:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_A^B H_z \, dz + \int_B^C H_\rho \, d\rho + \int_C^D H_{z,out} \, dz + \int_D^A H_\rho \, d\rho = I_{encl} = \frac{NI}{d} \Delta z$$

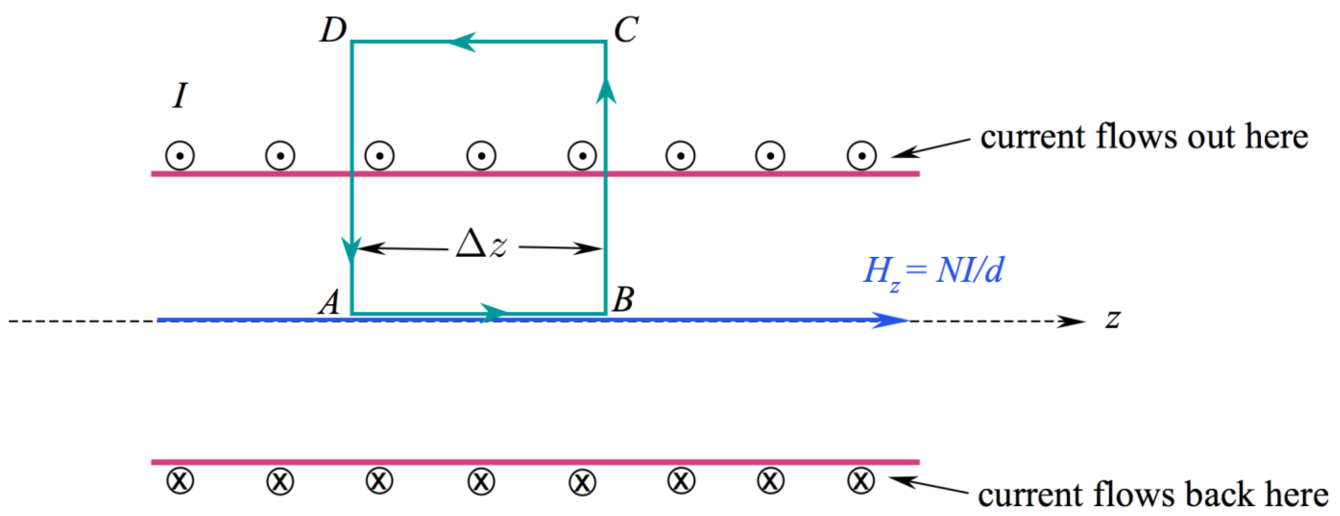
Where allowance is made for the existence of a radial H component, H_ρ



Radial Path Segments

The radial integrals will now cancel, because they are oppositely-directed, and because in the long coil, H_ρ is not expected to differ between the two radial path segments.

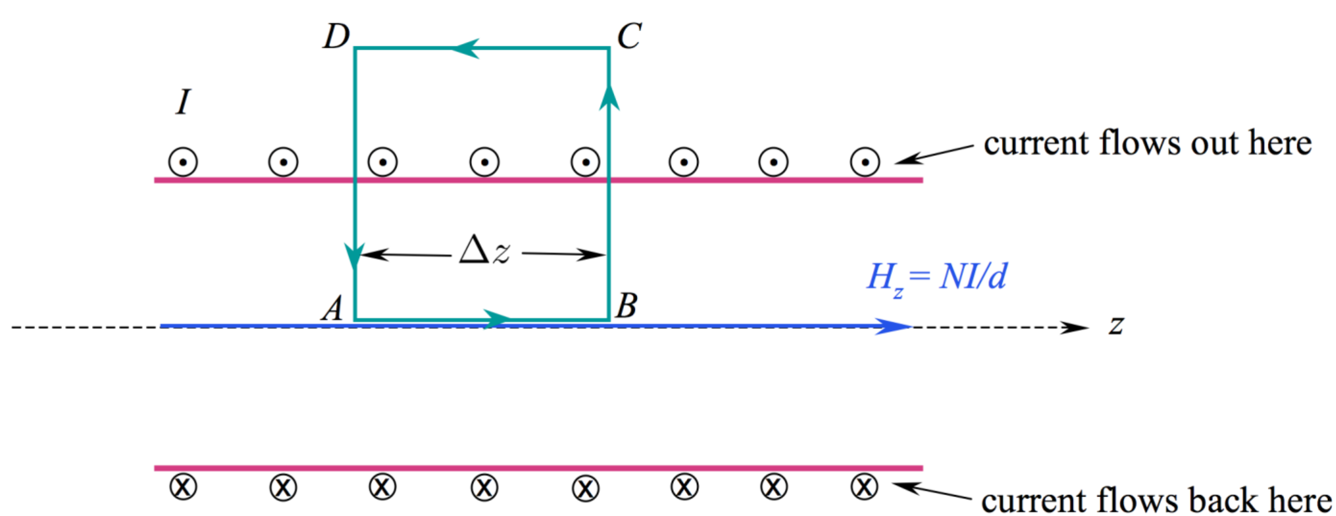
$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_A^B H_z \, dz + \int_B^C H_\rho \, d\rho + \int_C^D H_{z,out} \, dz + \int_D^A H_\rho \, d\rho = I_{encl} = \frac{NI}{d} \Delta z$$



Completing the Evaluation

What is left now are the two z integrations, the first of which we can evaluate as shown. Since this first integral result is equal to the enclosed current, it must follow that the second integral -- and the outside magnetic field -- are zero.

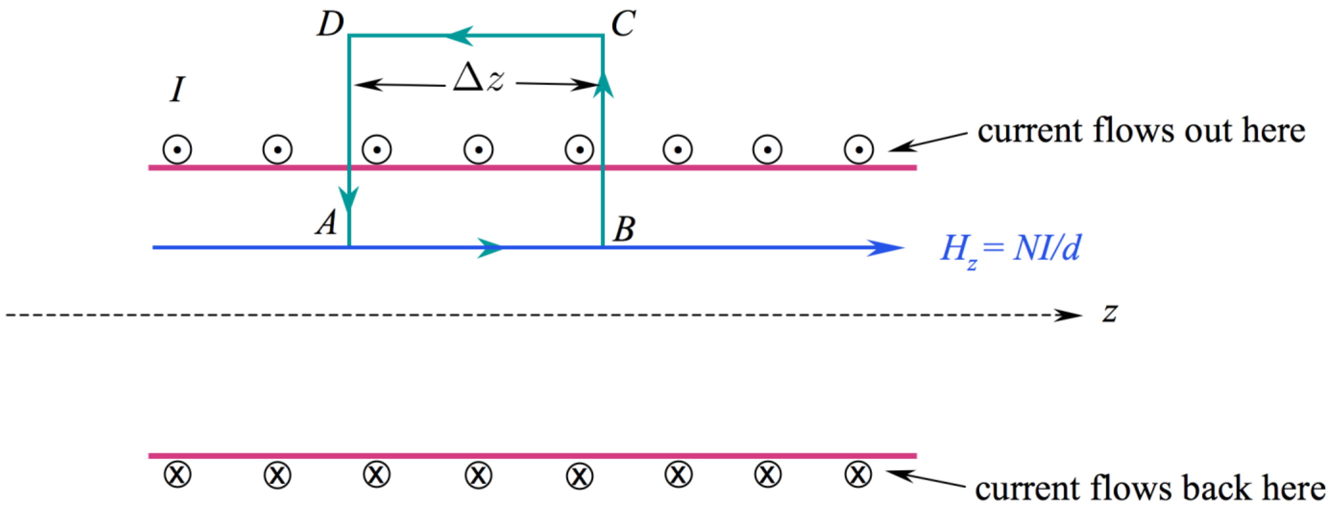
$$\oint \mathbf{H}\cdot d\mathbf{L} = \underbrace{\int_A^B H_z\,dz}_{(NI/d)\Delta z} + \int_C^D H_{z,out}\,dz = I_{encl} = \frac{NI}{d}\Delta z$$



Finding the Off-Axis Field

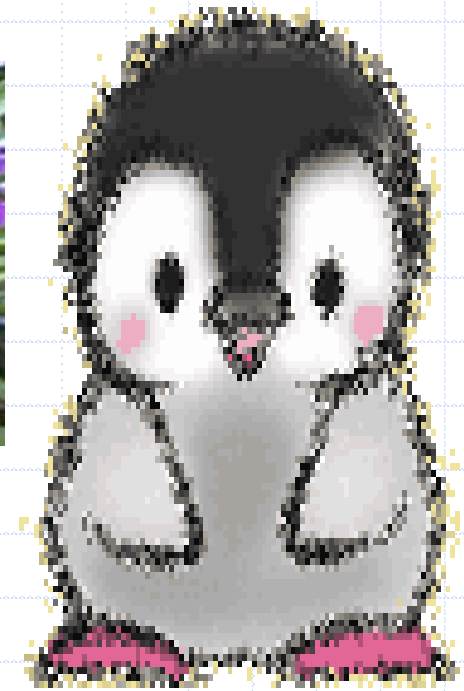
The situation does not change if the lower *z*-directed path is raised above the *z* axis. The vertical paths still cancel, and the outside field is still zero. The field along the path *A* to *B* is therefore *NI/d* as before.

$$\oint \mathbf{H} \cdot d\mathbf{L} = \underbrace{\int_A^B H_z \, dz}_{(NI/d)\Delta z} + \int_C^D H_{z,out} \, dz = I_{encl} = \frac{NI}{d} \Delta z$$



Conclusion: The magnetic field within a long solenoid is approximately constant throughout the coil cross-section, and is $H_z = NI/d$.

Thank you



Have a nice day.