Lecture 3

The Steady Magnetic Field

Electromagnetic Field Theory

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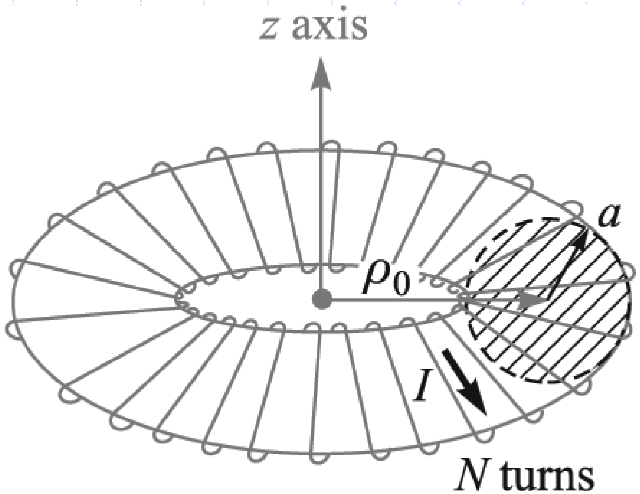
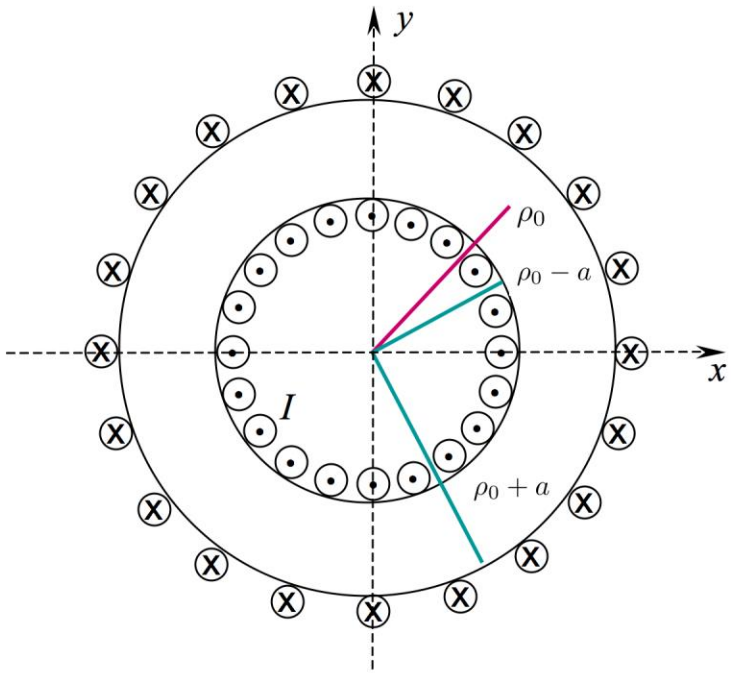
Toroid Magnetic Field

A toroid is a doughnut-shaped set of windings around a core material. The cross-section could be circular (as shown here, with radius *a*) or any other shape.

Below, a slice of the toroid is shown, with current emerging from the screen around the inner periphery (in the positive *z* direction). The windings are modeled as *N* individual current loops, each of which carries current *I*.





Ampere's Law as Applied to a Toroid

Ampere's Circuital Law can be applied to a toroid by taking a closed loop integral around the circular contour C at radius ρ. Magnetic field **H** is presumed to be circular, and a function of radius only at locations within the toroid that are not too close to the individual windings. Under this condition, we would assume:

$$\mathbf{H} = H_\phi \, \mathbf{a}_\phi$$

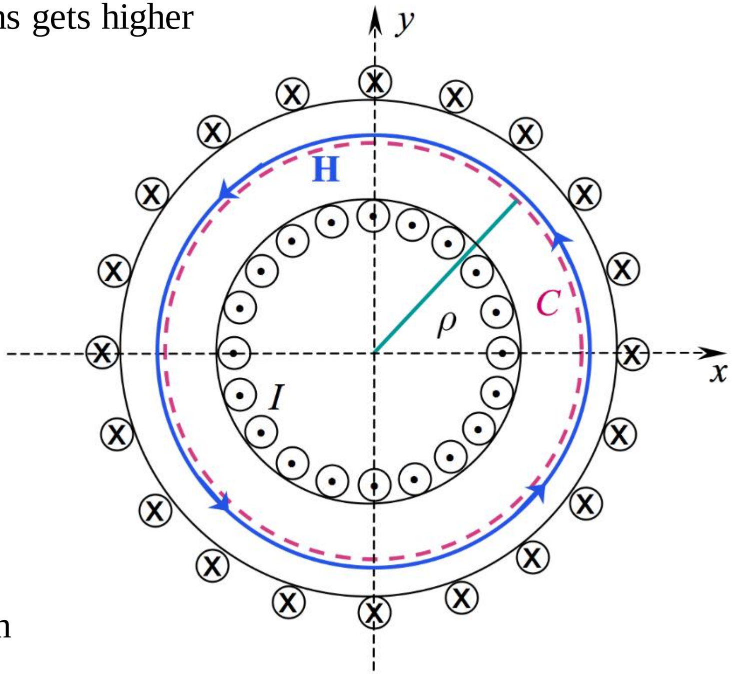
This approximation improves as the density of turns gets higher (using more turns with finer wire).

Ampere's Law now takes the form:

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho \, H_\phi = I_{encl} = NI$$

so that….

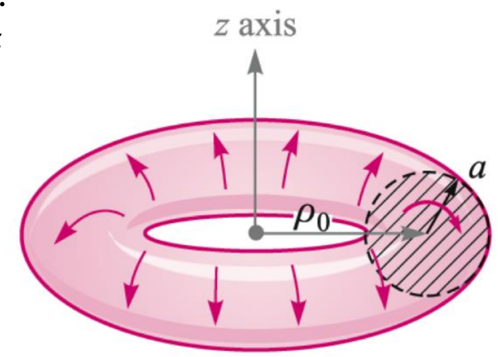
$$H_\phi = \frac{NI}{2\pi\rho} \quad (\rho_0 - a < \rho < \rho_0 + a)$$



Performing the same integrals over contours drawn in the regions $\rho < \rho_0 - a$ or $\rho > \rho_0 + a$ will lead to *zero* magnetic field there, because no current is enclosed in either case.

Surface Current Model of a Toroid

Consider a sheet current molded into a doughnut shape, as shown. The current density at radius $\rho_0 - a$ crosses the *xy* plane in the *z* direction and is given in magnitude by K_a



Ampere's Law applied to a circular contour *C* inside the toroid (as in the previous example) will take the form:

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = I_{encl} = 2\pi(\rho_0 - a)K_a$$

leading to…

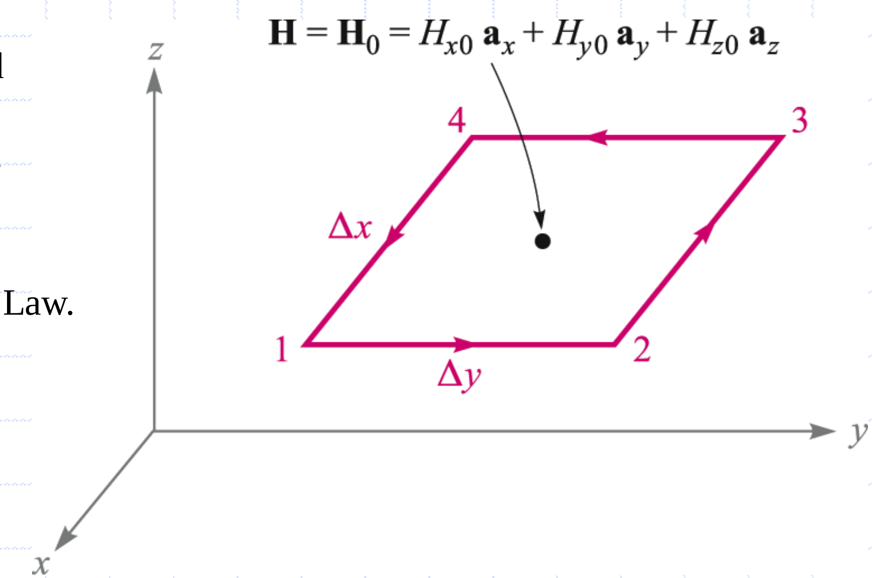
$$H_\phi = \frac{\rho_0 - a}{\rho} K_a$$

inside the toroid…. and the field is zero outside as before.

Ampere's Law as Applied to a Small Closed Loop.

Consider magnetic field **H** evaluated at the point shown in the figure. We can approximate the field over the closed path 1234 by making appropriate adjustments in the value of **H** along each segment.

The objective is to take the closed path integral and ultimately obtain the point form of Ampere's Law.



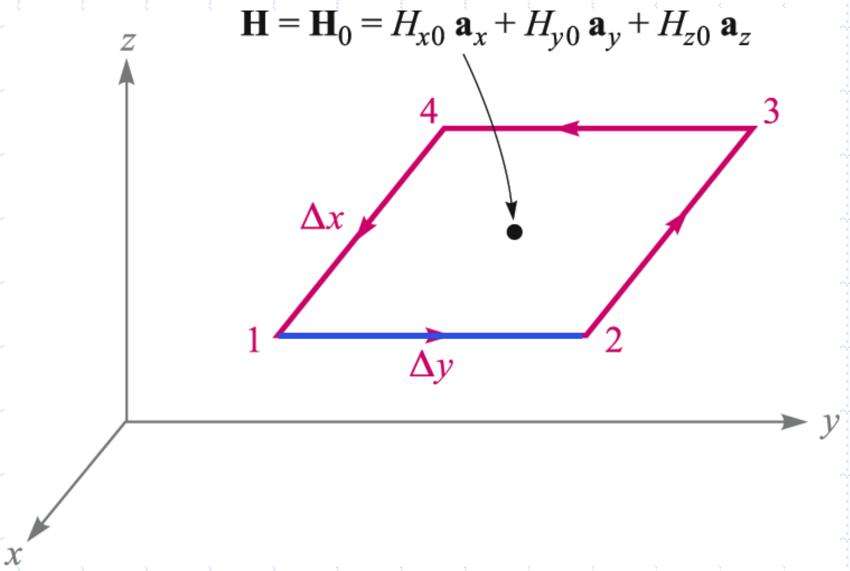
Approximation of **H** Along One Segment

Along path 1-2, we may write:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

where:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$



And therefore:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

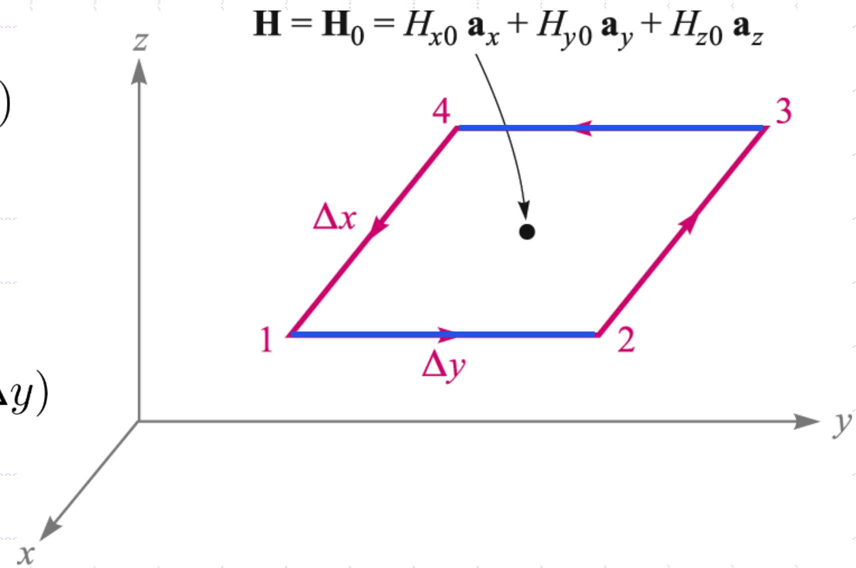
Contributions of *y*-Directed Path Segments

The contributions from the front and back sides will be:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)(\Delta y)$$

The contribution from the opposite side is:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{3-4} \doteq \left(H_{y0} - \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)(-\Delta y)$$



Note the path directions as specified in the figure, and how these determine the signs used .

This leaves the left and right sides…..

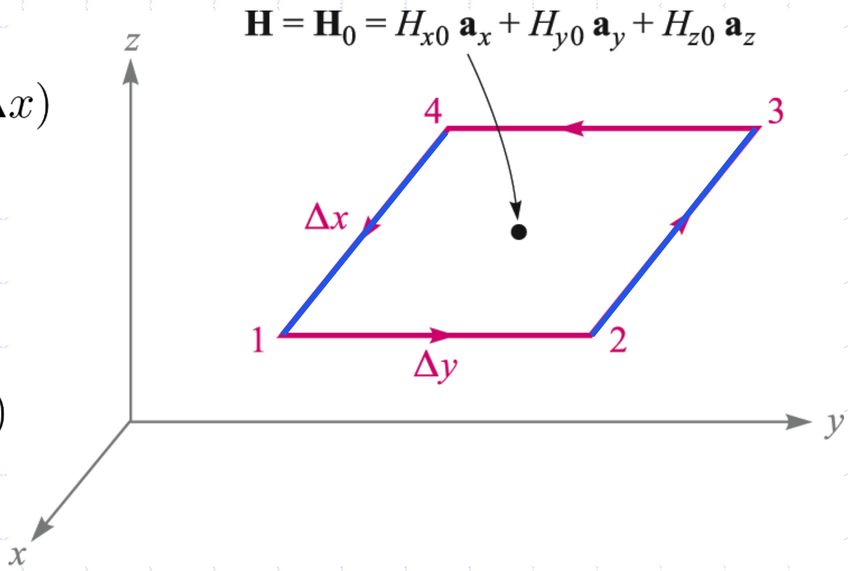
Contributions of *x*-Directed Path Segments

Along the right side (path 2-3):

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{2-3} \doteq \left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)(-\Delta x)$$

…and the contribution from the left side (path 4-1) is:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{4-1} \doteq \left(H_{x0} - \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)(\Delta x)$$



The next step is to add the contributions of all four sides to find the closed path integral:

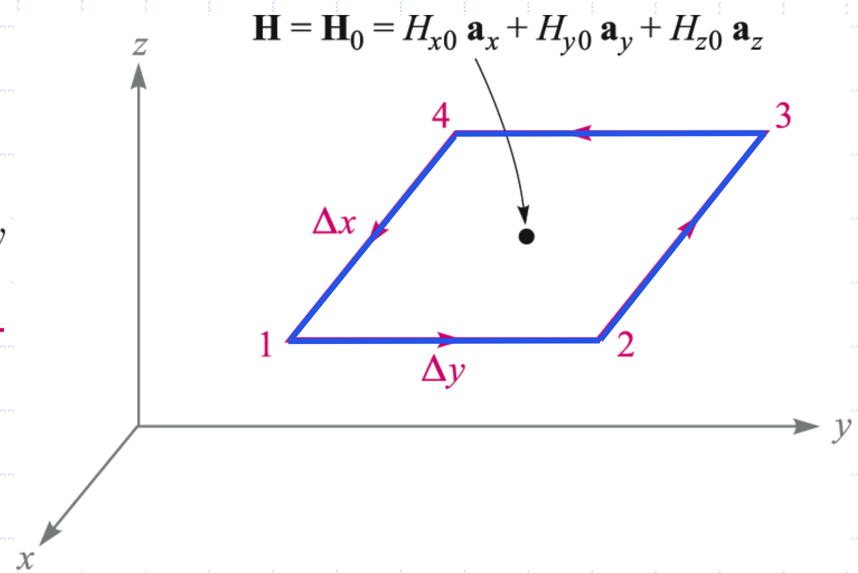
Net Closed Path Integral

The total integral will now be the sum:

$$\oint \mathbf{H}\cdot d\mathbf{L} \doteq (\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} + (\mathbf{H}\cdot\Delta\mathbf{L})_{2-3} + (\mathbf{H}\cdot\Delta\mathbf{L})_{3-4} + (\mathbf{H}\cdot\Delta\mathbf{L})_{4-1}$$

and using our previous results, the becomes:

$$\oint \mathbf{H}\cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\,\Delta y$$



Relation to the Current Density

By Ampere's Law, the closed path integral of **H** is equal to the enclosed current, approximated in this case by the current density at the center, multiplied by the loop area:

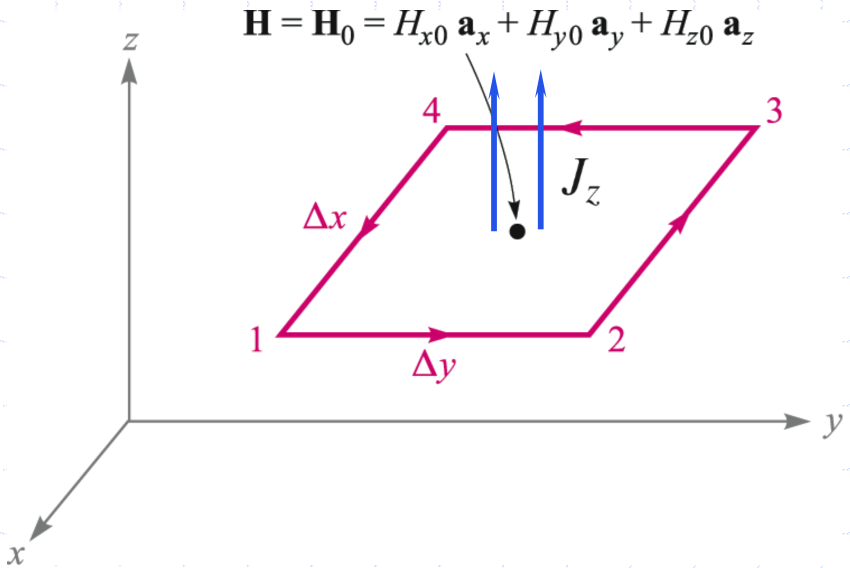
$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y \doteq J_z \Delta x \Delta y$$

Dividing by the loop area, we now have:

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

The expression becomes exact as the loop area approaches zero:

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$



Other Loop Orientations

The same exercise can be carried with the rectangular loop in the other two orthogonal orientations. The results are:

Loop in *yz* plane:

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

Loop in *xz* plane:

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

Loop in *xy* plane:

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

This gives all three components of the current density field.

Curl of a Vector Field

The previous exercise resulted in the rectangular coordinate representation of the *Curl* of **H**.

In general, the curl of a vector field is another field that is normal to the original field.

The curl component in the direction *N*, normal to the plane of the integration loop is:

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}$$

where ΔS_N is the planar area enclosed by the closed line integral.

The direction of *N* is taken using the right-hand convention: With fingers of the right hand oriented in the direction of the path integral, the thumb points in the direction of the normal (or curl).

Curl in Rectangular Coordinates

Assembling the results of the rectangular loop integration exercise, we find the vector field that comprises curl **H**:

$$\text{curl}\,\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

An easy way to calculate this is to evaluate the following determinant:

$$\text{curl}\,\mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

which we see is equivalent to the cross product of the del operator with the field:

$$\text{curl}\,\mathbf{H} = \nabla \times \mathbf{H}$$

Curl in Other Coordinate Systems

…a little more complicated!

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)}$$

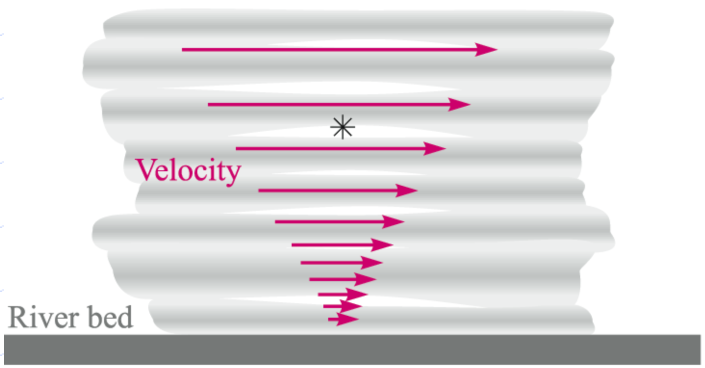
$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)}$$

Look these up as needed….

Visualization of Curl

Consider placing a small "paddle wheel" in a flowing stream of water, as shown below. The wheel axis points into the screen, and the water velocity decreases with increasing depth.

The wheel will rotate clockwise, and give a curl component that points into the screen (right-hand rule).



Positioning the wheel at all three orthogonal orientations will yield measurements of all three components of the curl. Note that the curl is directed normal to both the field and the direction of its variation.

Another Maxwell Equation

It has just been demonstrated that:

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z = \mathbf{J}$$

…..which is in fact one of Maxwell's equations for static fields:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

This is Ampere's Circuital Law in point form.

….and Another Maxwell Equation

We already know that for a *static* electric field:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

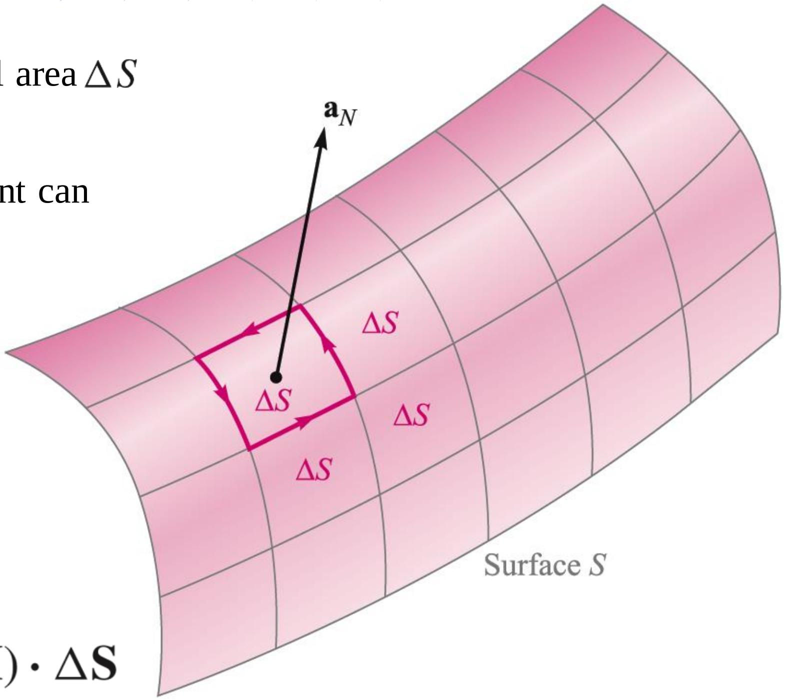
This means that: $$\boxed{\nabla \times \mathbf{E} = 0}$$ (applies to a static electric field)

Recall the condition for a conservative field: that is, its closed path integral is zero everywhere.

Therefore, a field is conservative if it has *zero curl* at all points over which the field is defined.

Curl Applied to Partitions of a Large Surface

Surface S is paritioned into sub-regions, each of small area ΔS



The curl component that is normal to a surface element can be written using the definition of curl:

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

or:

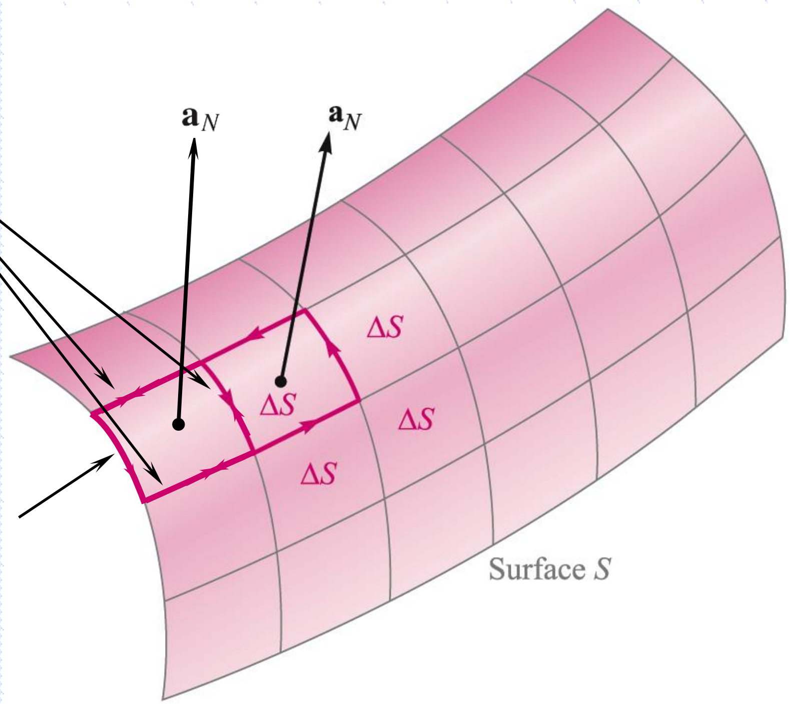
$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S}$$

We now apply this to every partition on the surface, and add the results….

Adding the Contributions

We now evaluate and add the curl contributions from all surface elements, and note that adjacent path integrals will all cancel!

This means that the only contribution to the overall path integral will be around the outer periphery of surface *S*.



Using our previous result, we now write:

$$\sum_{\substack{\text{all surface}\\ \text{elements}}} \oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq \sum_{\substack{\text{all surface}\\ \text{elements}}} \nabla \times \mathbf{H} \cdot \mathbf{a}_N \Delta S$$

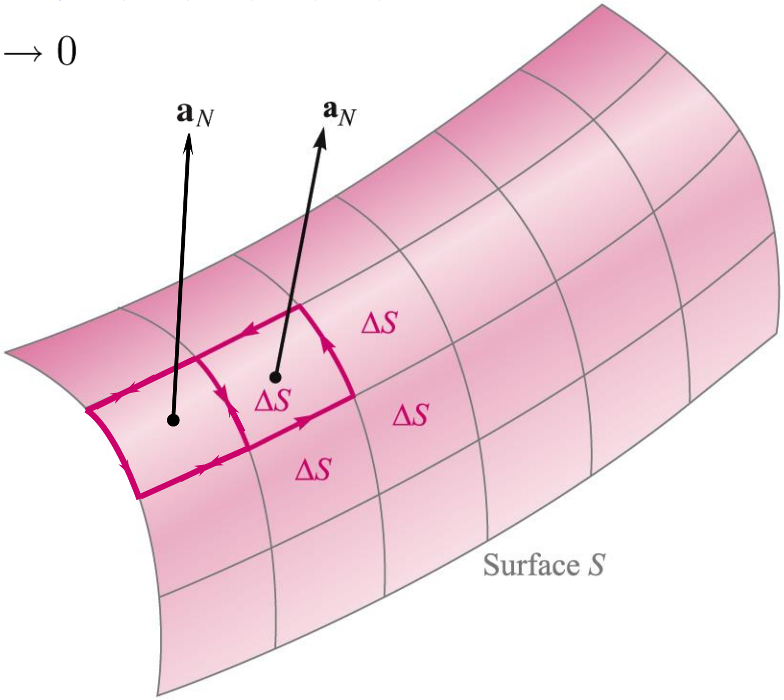
Stokes' Theorem

We now take our previous result, and take the limit as $\Delta S \rightarrow 0$

$$\sum_{\text{all surface elements}} \oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq \sum_{\text{all surface elements}} \nabla \times \mathbf{H} \cdot \mathbf{a}_N \Delta S$$

In the limit, this side becomes the path integral of **H** over the outer perimeter because all interior paths cancel

In the limit, this side becomes the integral of the curl of **H** over surface *S*



The result is Stokes' Theorem

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

This is a valuable tool to have at our disposal, because it gives us two ways to evaluate the same thing!

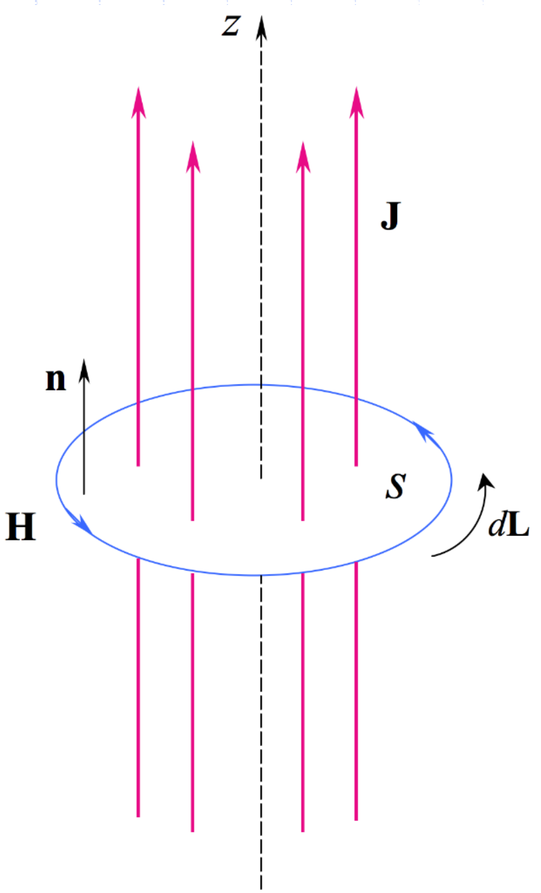
Obtaining Ampere's Circuital Law in Integral Form, using Stokes' Theorem

Begin with the point form of Ampere's Law for static fields:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Integrate both sides over surface S:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$



..in which the far right hand side is found from the left hand side using Stokes' Theorem. The closed path integral is taken around the perimeter of S. Again, note that we use the right-hand convention in choosing the direction of the path integral.

The center expression is just the net current through surface S, so we are left with the integral form of Ampere's Law:

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

Magnetic Flux and Flux Density

We are already familiar with the concept of electric flux:

$$\Psi = \int_s \mathbf{D} \cdot d\mathbf{S} \quad \text{Coulombs}$$

in which the electric flux density in free space is: $\mathbf{D} = \epsilon_0 \mathbf{E} \quad \mathrm{C/m^2}$

and where the free space permittivity is $\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F/m}$

In a similar way, we can define the *magnetic flux* in units of Webers (Wb):

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{S} \quad \text{Webers}$$

in which the *magnetic flux density* (or *magnetic* induction) in free space is: $\mathbf{B} = \mu_0 \mathbf{H} \quad \mathrm{Wb/m^2}$

and where the free space *permeability* is $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H/m}$

This is a *defined* quantity, having to do with the definition of the ampere (we will explore this later).

A Key Property of **B**

If the flux is evaluated through a closed surface, we have in the case of electric flux, Gauss' Law:

$$\Psi_{net} = \oint_s \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

If the same were to be done with magnetic flux density, we would find:

$$\Phi_{net} = \oint_s \mathbf{B} \cdot d\mathbf{S} = 0$$

The implication is that (for our purposes) there are no magnetic charges
-- specifically, *no point sources of magnetic field exist*. A hint of this has already been observed, in that magnetic field lines always close on themselves.

Another Maxwell Equation

We may rewrite the closed surface integral of **B** using the divergence theorem, in which the right hand integral is taken over the volume surrounded by the closed surface:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \; dv = 0$$

Because the result is zero, it follows that

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

This result is known as Gauss' Law for the magnetic field in point form.

Maxwell's Equations for Static Fields

We have now completed the derivation of Maxwell's equations for no time variation. In point form, these are:

$$\nabla \cdot \mathbf{D} = \rho_v$$ Gauss' Law for the electric field

$$\nabla \times \mathbf{E} = 0$$ Conservative property of the static electric field

$$\nabla \times \mathbf{H} = \mathbf{J}$$ Ampere's Circuital Law

$$\nabla \cdot \mathbf{B} = 0$$ Gauss' Law for the Magnetic Field

where, in free space:

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

Significant changes in the above four equations will occur when the fields are allowed to vary with time, as we'll see later.

Maxwell's Equations in Large Scale Form

The divergence theorem and Stokes' theorem can be applied to the previous four point form equations to yield the integral form of Maxwell's equations for static fields:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_\nu d\nu$$

Gauss' Law for the electric field

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Conservative property of the static electric field

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

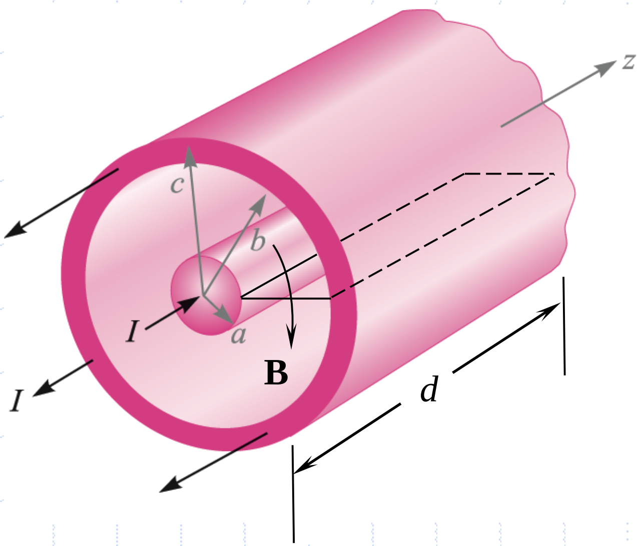
Ampere's Circuital Law

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Gauss' Law for the magnetic field

Example: Magnetic Flux Within a Coaxial Line

Consider a length d of coax, as shown here. The magnetic field strength between conductors is:



$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and so: $$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

The magnetic flux is now the integral of **B** over the flat surface between radii a and b, and of length d along z:

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

The result is: $$\Phi = \frac{\mu_0 I d}{2\pi} \ln\frac{b}{a}$$

The coax line thus "stores" this amount of magnetic flux in the region between conductors. This will have importance when we discuss inductance in a later lecture.

Scalar Magnetic Potential

We are already familiar with the relation between the scalar electric potential and electric field:

$$\mathbf{E} = -\nabla V$$

So it is tempting to define a scalar magnetic potential such that:

$$\mathbf{H} = -\nabla V_m$$

This rule must be consistent with Maxwell's equations, so therefore:

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

But the curl of the gradient of any function is identically zero! Therefore, the scalar magnetic potential is valid only in regions where the current density is zero (such as in free space).

So we define scalar magnetic potential with a condition:

$$\mathbf{H} = -\nabla V_m \qquad (\mathbf{J} = 0)$$

Further Requirements on the Scalar Magnetic Potential

The other Maxwell equation involving magnetic field must also be satisfied. This is:

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0$$ in free space

Therefore: $$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

..and so the scalar magnetic potential satisfies Laplace's equation (again with the restriction that current density must be zero:

$$\nabla^2 V_m = 0 \quad (\mathbf{J} = 0)$$

Example: Coaxial Transmission Line

With the center conductor current flowing out of the screen, we have

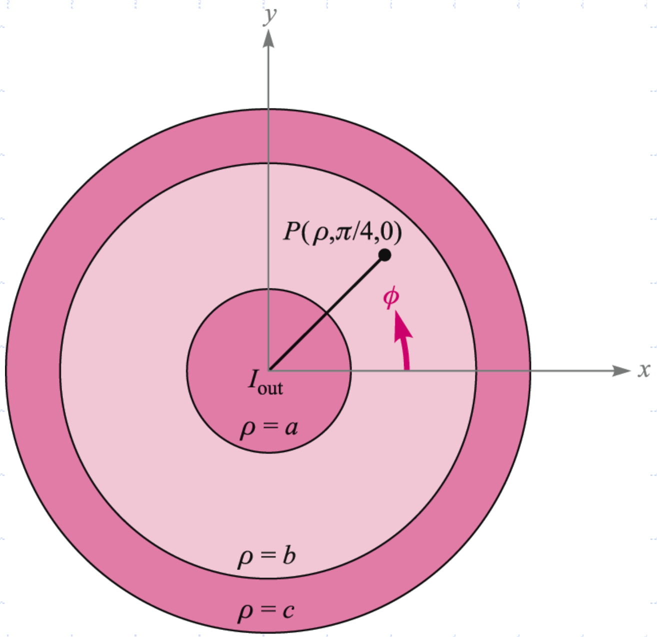
$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

Thus: $$\frac{I}{2\pi\rho} = -\nabla V_m\Big|_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

So we solve: $$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

.. and obtain: $$V_m = -\frac{I}{2\pi}\phi$$

where the integration constant has been set to zero



Ambiguities in the Scalar Potential

The scalar potential is now:

$$V_m = -\frac{I}{2\pi}\phi$$

where the potential is zero at $\phi = 0$

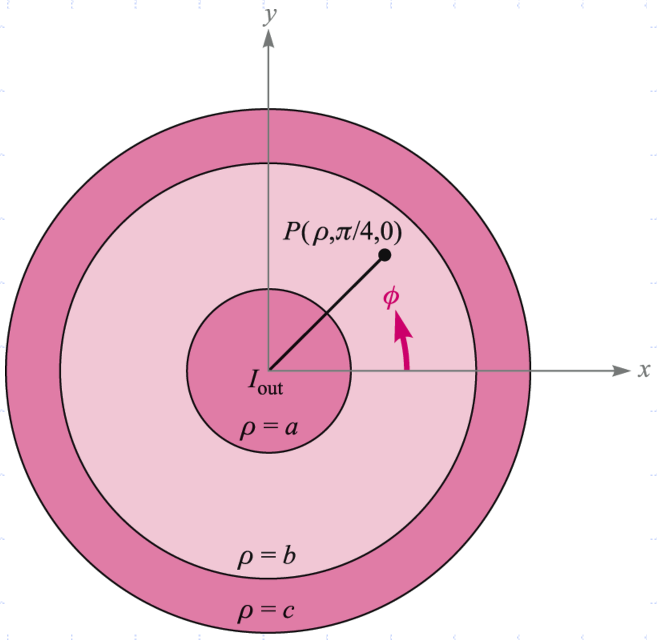
At point *P* ($\phi = \pi/4$) the potential is

$$V_{mP}(\phi = \pi/4) = -I/8$$

But wait! As ϕ increases to $\phi = 2\pi$
we have returned to the same physical location, and
the potential has a new value of -*I*.

In general, the potential at *P* will be multivalued, and will acquire a new value after each full rotation in the *xy* plane:

$$V_{mP} = \frac{I}{2\pi}\left(2n - \tfrac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$



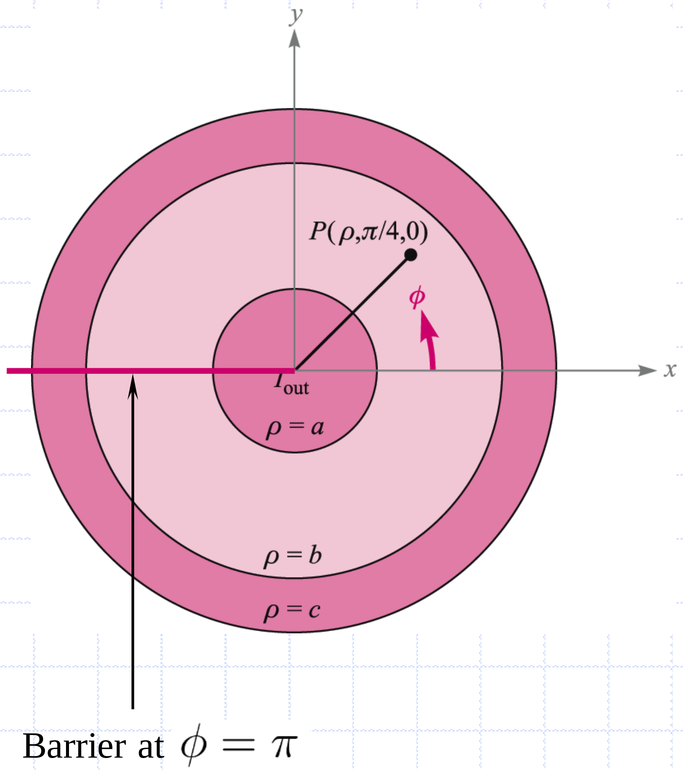
Overcoming the Ambiguity

To remove the ambiguity, we construct a mathematical barrier at any value of phi. The angle domain cannot cross this barrier in either direction, and so the potential function is restricted to angles on either side. In the present case we choose the barrier to lie at $\phi = \pi$ so that

$$V_m = -\frac{I}{2\pi}\phi \quad (-\pi < \phi < \pi)$$

The potential at point P is now single-valued:

$$V_{mP} = -\frac{I}{8} \quad \left(\phi = \frac{\pi}{4}\right)$$



Vector Magnetic Potential

We make use of the Maxwell equation: $\nabla \cdot \mathbf{B} = 0$

.. and the fact that the divergence of the curl of any vector field is identically zero (show this!)

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

This leads to the definition of the *magnetic vector potential*, **A**:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Thus: $\mathbf{H} = \dfrac{1}{\mu_0} \nabla \times \mathbf{A}$

and Ampere's Law becomes $\nabla \times \mathbf{H} = \mathbf{J} = \dfrac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$

Equation for the Vector Potential

We start with: $$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$$

Then, introduce a vector identity that defines the *vector Laplacian*:

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Using a (lengthy) procedure (see Sec. 7.7) it can be proven that $\nabla \cdot \mathbf{A} = 0$

- We are therefore left with

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

The Direction of **A**

We now have $$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

In rectangular coordinates:

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

(not so simple in the other coordinate systems)

The equation separates to give:

$$\nabla^2 A_x = -\mu_0 J_x$$

$$\nabla^2 A_y = -\mu_0 J_y$$

$$\nabla^2 A_z = -\mu_0 J_z$$

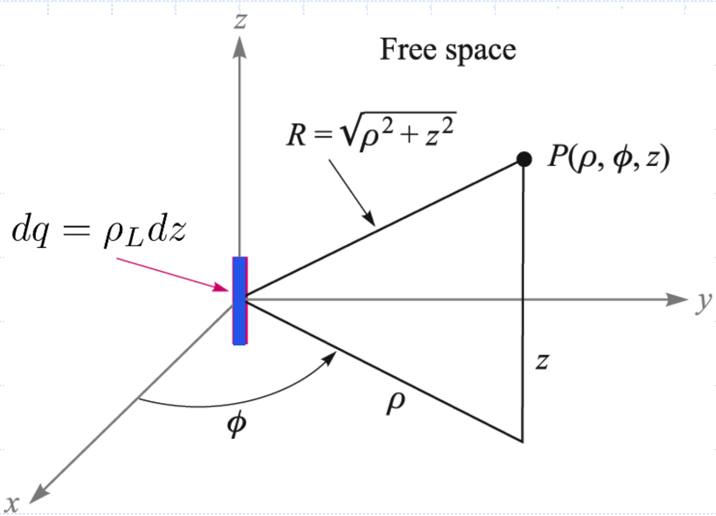
This indicates that the direction of **A** will be the same as that of the current to which it is associated.

The vector field, **A**, existing in all space, is sometimes described as being a “fuzzy image” of its generating current.

Expressions for Potential

Consider a differential elements, shown here. On the left is a point charge represented by a differential length of line charge. On the right is a differential current element. The setups for obtaining potential are identical between the two cases.

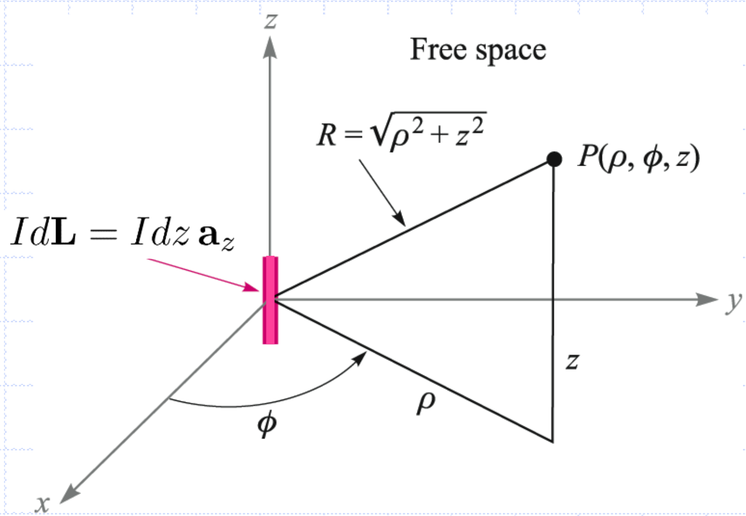
Line Charge



Scalar Electrostatic Potential

$$dV = \frac{dq}{4\pi\epsilon_0 R} = \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

Line Current



Vector Magnetic Potential

$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R} = \frac{\mu_0 I dz\,\mathbf{a}_z}{4\pi R}$$

General Expressions for Vector Potential

For large scale charge or current distributions, we would sum the differential contributions by integrating over the charge or current, thus:

$$V = \int \frac{\rho_L \, dL}{4\pi\epsilon_0 R}$$

and

$$\mathbf{A} = \oint \frac{\mu_0 I \, d\mathbf{L}}{4\pi R}$$

The closed path integral is taken because the current must close on itself to form a complete circuit.

For surface or volume current distributions, we would have, respectively:

$$\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K} \, dS}{4\pi R}$$

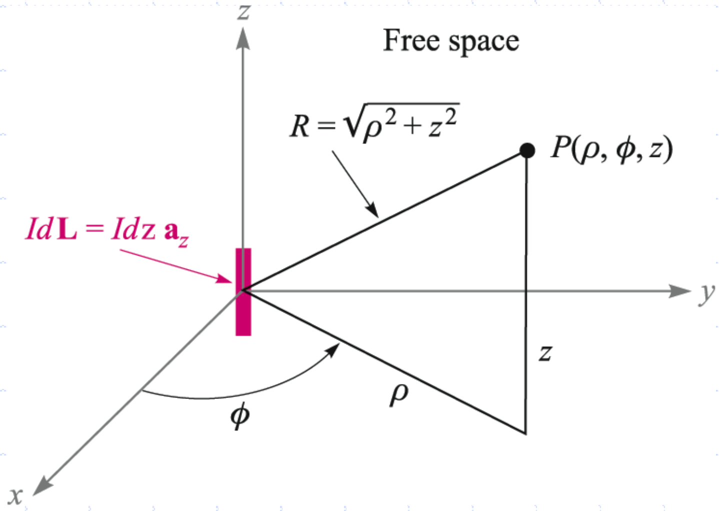
or

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J} \, dv}{4\pi R}$$

in the same manner that we used for scalar electric potential.

Example

We continue with the differential current element as shown here:



In this case

$$dA = \frac{\mu_0 I\, d\mathbf{L}}{4\pi R}$$

becomes at point *P*:

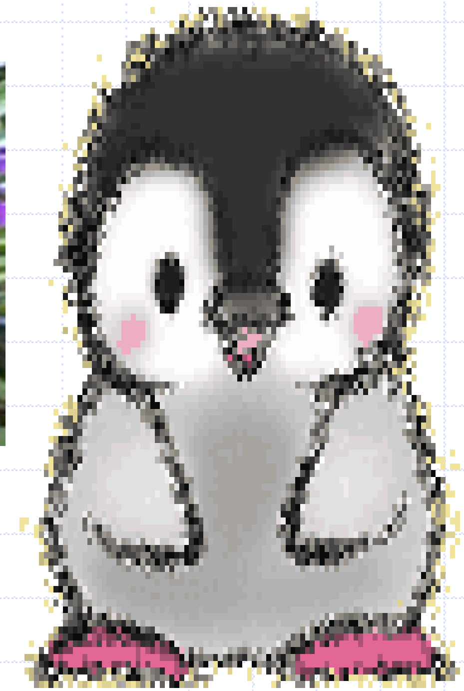
$$d\mathbf{A} = \frac{\mu_0 I\, dz\, \mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

Now, the curl is taken in cylindrical coordinates:

$$d\mathbf{H} = \frac{1}{\mu_0}\nabla \times d\mathbf{A} = \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\mathbf{a}_\phi = \frac{I\,dz}{4\pi}\frac{\rho}{(\rho^2 + z^2)^{3/2}}\mathbf{a}_\phi$$

This is the same result as found using the Biot-Savart Law (as it should be)

Thank you



Have a nice day .