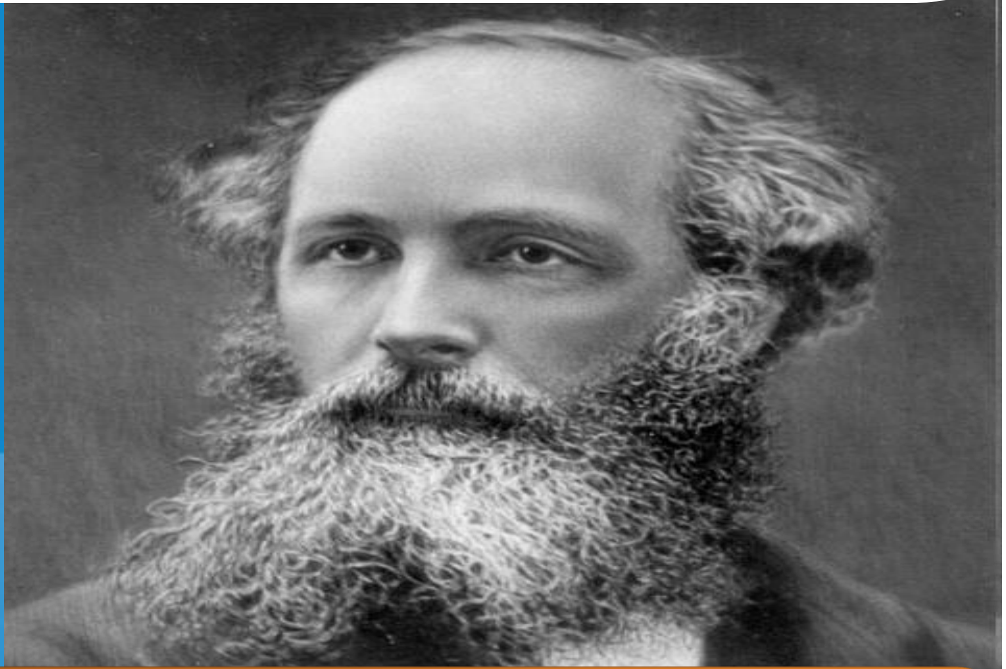
Lecture 7

The Uniform Plane Wave

Electromagnetic Field Theory



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Phasor Form

- Taking the **partial derivative** of any field quantity **w. r. t. time** is equivalent to multiplying the corresponding phasor by $j\omega$.
- We can express Eq. (8) $\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t}$ (using sinusoidal fields) as

$$\frac{\partial \mathcal{H}_y}{\partial z} = -\epsilon_0 \frac{\partial \mathcal{E}_x}{\partial t} \qquad (20)$$

- where, in a manner consistent with (19):

$$\mathcal{E}_x(z,t) = \frac{1}{2} E_{xs}(z)\, e^{j\omega t} + c.c. \quad \text{and} \quad \mathcal{H}_y(z,t) = \frac{1}{2} H_{ys}(z)\, e^{j\omega t} + c.c. \qquad (21)$$

- On substituting the fields in (21) into (20), the latter equation simplifies to

$$\frac{dH_{ys}(z)}{dz} = -j\omega\epsilon_0 E_{xs}(z) \qquad (22)$$

Phasor Form

- For sinusoidal or cosinusoidal time variation,

$$\mathbf{E} = E_x \mathbf{a}_x \qquad E_x = E(x,y,z)\cos(\omega t + \psi)$$

- Using Euler's Identity $e^{j\omega t} = \cos\omega t + j\sin\omega t$
- The field is $E_x = \text{Re}\left[E(x,y,z)e^{j\psi}e^{j\omega t}\right]$
- The field in the phasor form is $E_{xs} = E(x,y,z)e^{j\psi}$ $\quad \mathbf{E}_s = E_{xs}\mathbf{a}_x$
- For any sinusoidal time variation for field $\frac{\partial}{\partial t}\vec{E} = j\omega\vec{E}$ and $\frac{\partial}{\partial t}\vec{H} = j\omega\vec{H}$

$$\frac{\partial}{\partial t} \to jw$$

- The Maxwell's equation in phasor form

$$\nabla \times \mathbf{H}_s = j\omega\epsilon_0\mathbf{E}_s \qquad (23)$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu_0\mathbf{H}_s \qquad (24)$$

$$\nabla \cdot \mathbf{E}_s = 0 \qquad (25)$$

$$\nabla \cdot \mathbf{H}_s = 0 \qquad (26)$$

- Eqs. (23) through (26) may be used to obtain the **sinusoidal steady-state vector** form of the **wave equation in free space**.

Wave Equation

- Using Maxwell's Equation the steady state wave equation can be written as:

$$\nabla\times(\nabla\times \mathrm{E}_s) = \nabla(\nabla\cdot \mathrm{E}_s) - \nabla^2 \mathrm{E}_s = -j\omega\mu_o \nabla\times \mathrm{H}_s = \omega^2\mu_o\varepsilon_o \mathrm{E}_s$$

- Since $\nabla\cdot \mathrm{E}_s = 0 \Rightarrow \nabla^2 \mathrm{E}_s = -k_o^2 \mathrm{E}_s$ — Vector Helmholtz Wave Equation

k_o, the free space wavenumber, $k_o = \omega\sqrt{\mu_o\varepsilon_o}$

- The wave equation for the x-components is $\nabla^2 \mathrm{E}_{xs} = -k_o^2 \mathrm{E}_{xs}$
- Using the definition of Del operator

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} = -k_o^2 E_{xs}$$

- The field components does not change with x or y so the wave equation is lead to ordinary differential equation

$$\frac{d^2 E_{xs}}{dz^2} = -k_o^2 E_{xs}$$

- the solution of which

$$E_{xs}(z) = E_{x0}e^{-jk_0 z} + E'_{x0}e^{jk_0 z} \quad (31)$$

WAVE PROPAGATION IN FREE SPACE

- Given $\mathbf{E}_s$, $\mathbf{H}_s$ is most easily obtained from (24):

$$\nabla \times \mathbf{E}_s = -j\omega\mu_0 \mathbf{H}_s$$

- which is greatly simplified for a single E_{xs} component varying only with z,

$$\frac{dE_{xs}}{dz} = -j\omega\mu_0 H_{ys}$$

- Using (31) for $E_{xs} = E_{x0}e^{-jk_0z} + E'_{x0}e^{jk_0z}$, we have

$$H_{ys} = -\frac{1}{j\omega\mu_0}\left[(-jk_0)E_{x0}e^{-jk_0z} + (jk_0)E'_{x0}e^{jk_0z}\right]$$

$$= E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{-jk_0z} - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{jk_0z} = H_{y0}e^{-jk_0z} + H'_{y0}e^{jk_0z} \qquad (32)$$

- In real instantaneous form, this becomes

$$H_y(z,t) = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t - k_0 z) - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t + k_0 z) \qquad (33)$$

- where E_{x0} and E'_{x0} are assumed real.

WAVE PROPAGATION IN FREE SPACE

- In general, we find from (32) that the electric and magnetic field amplitudes of the forward-propagating wave in free space are related through

$$E_{x0} = \sqrt{\frac{\mu_0}{\epsilon_0}} H_{y0} = \eta_0 H_{y0}$$

- We also find the backward-propagating wave amplitudes are related through

$$E'_{x0} = -\sqrt{\frac{\mu_0}{\epsilon_0}} H'_{y0} = -\eta_0 H'_{y0}$$

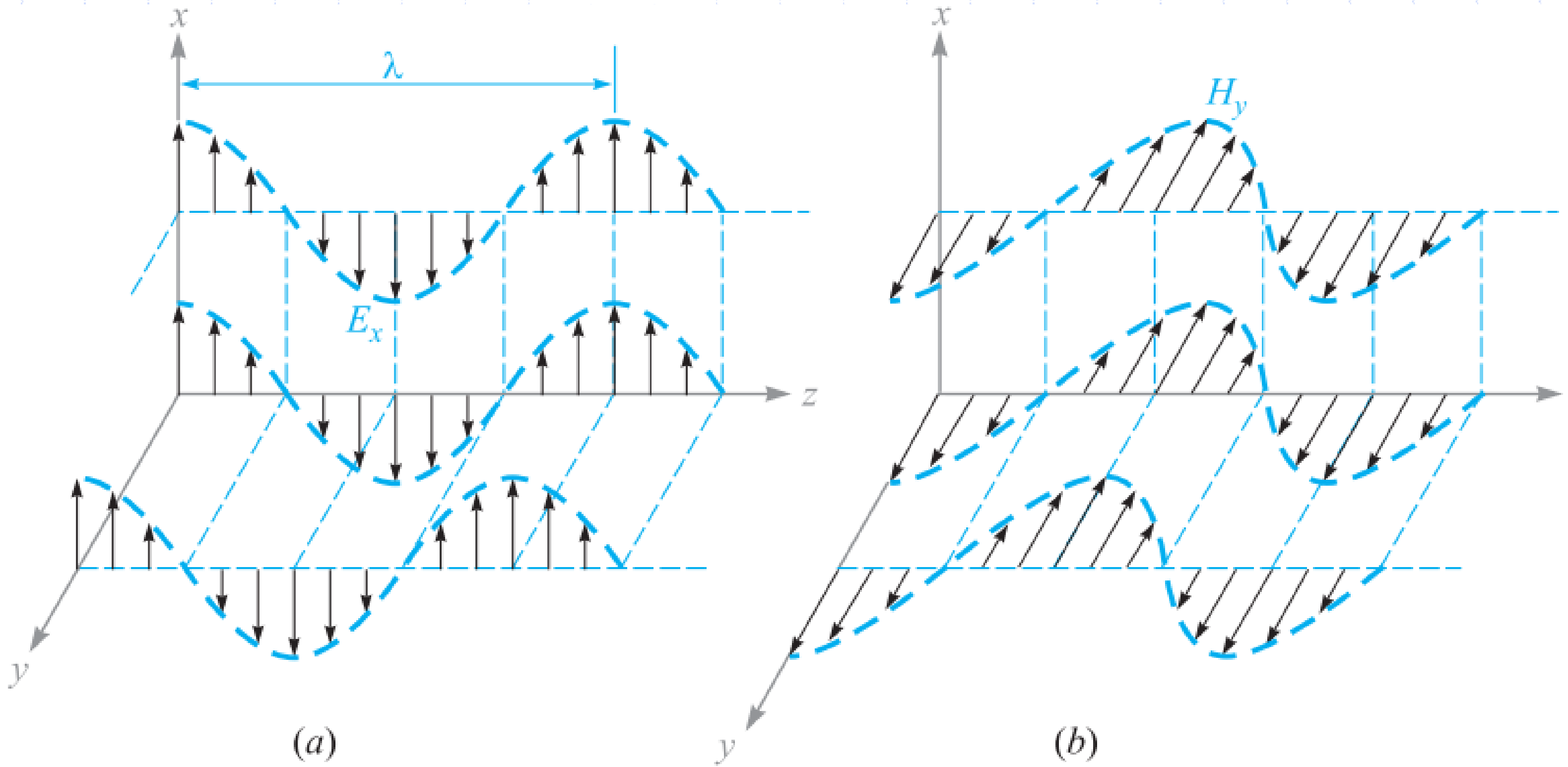
- where the intrinsic impedance of free space is defined as

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \doteq 120\pi \ \Omega$$

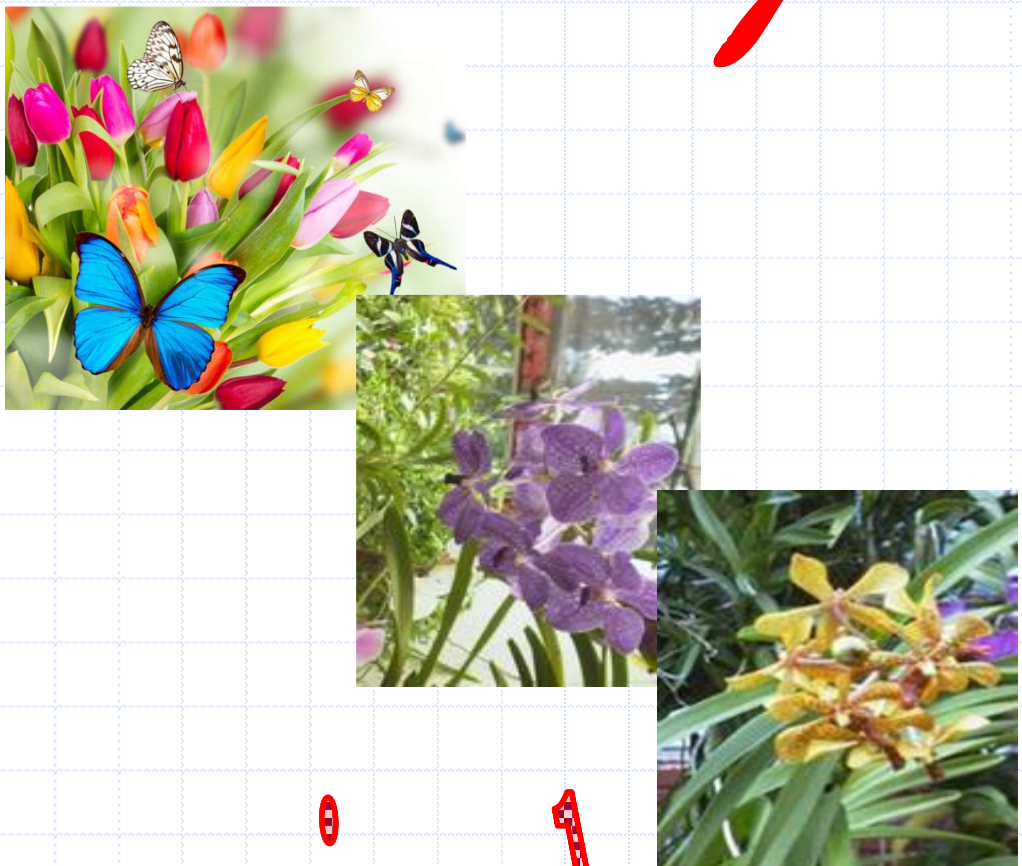
- The dimension of η_0 in ohms is immediately evident from its definition as the ratio of E (in units of V/m) to H (in units of A/m).

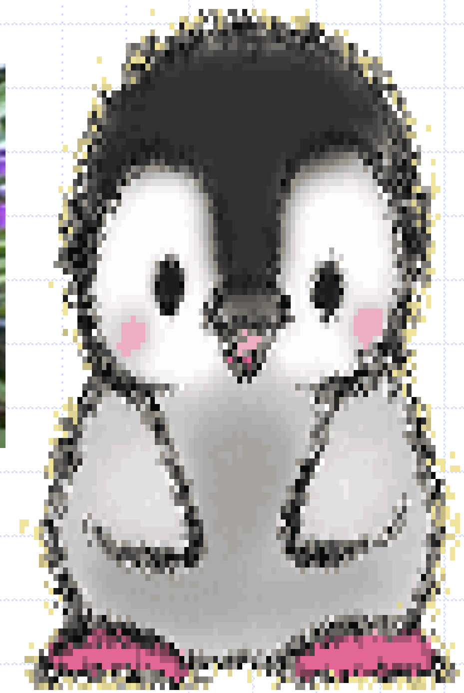
The fields

- The uniform plane cannot exists because it extends to infinity in two dimensions at least and represents an infinite a mount of energy.
- The far field of the antenna can be represent by a uniform plane wave



Thank you





Have a nice day.