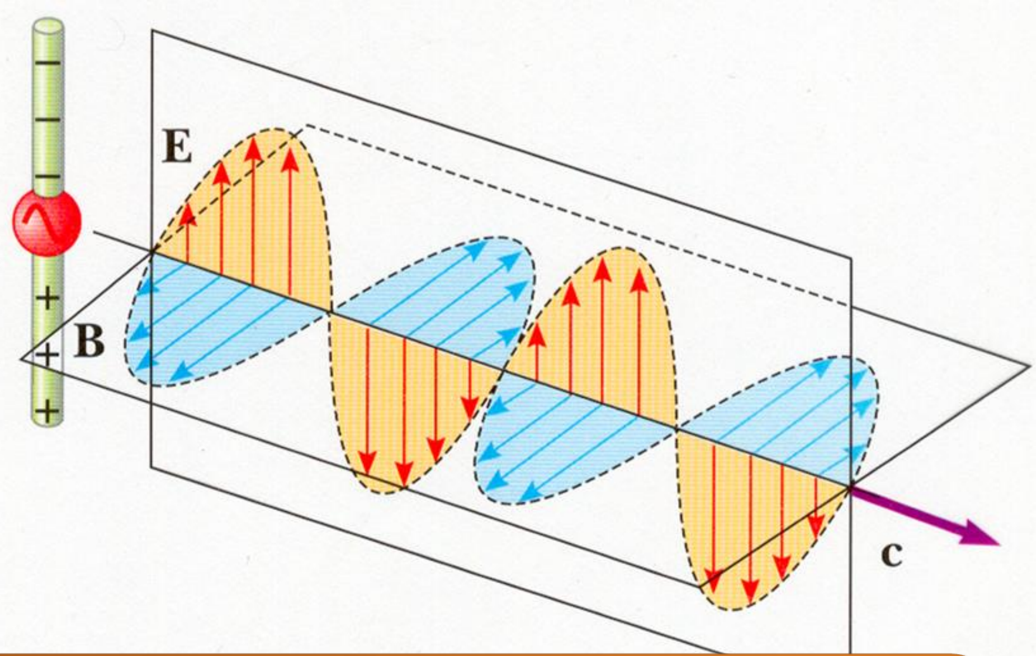
Lecture 8

The Uniform Plane Wave III

Electromagnetic Field Theory



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The Poynting vector and Power

- Start from Maxwell's equation

$$\nabla \times \mathbf{H} = \mathrm{J} + \frac{\partial \mathbf{D}}{\partial t}$$

- Next, take the scalar product of both sides with E,

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

- We then introduce the following vector identity:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \nabla \times \mathbf{H} + \mathbf{H} \cdot \nabla \times \mathbf{E}$$

- Then,

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

- where $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
- Therefore

$$-\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

or

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t}$$

The Poynting vector and Power

- The two time derivatives can be **rearranged** as follows:

$$\epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) \quad \text{and} \quad \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right)$$

- With these,

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right)$$

- Finally, we integrate throughout a volume:

$$-\int_{\text{vol}} \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = \int_{\text{vol}} \mathbf{J} \cdot \mathbf{E}\, dv + \int_{\text{vol}} \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) dv + \int_{\text{vol}} \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) dv$$

- Applying the divergence theorem to the LHS, thus converting the volume integral into an integral over the surface that encloses the volume. On the RHS, the operations of spatial integration and time differentiation are **interchanged**.

$$-\oint_{\text{area}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_{\text{vol}} \mathbf{J} \cdot \mathbf{E}\, dv + \frac{d}{dt}\int_{\text{vol}} \left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) dv + \frac{d}{dt}\int_{\text{vol}} \left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) dv$$

This equation is known as Poynting's theorem.

The Poynting vector and Power

- $\int_{\text{vol}} \mathbf{J}\cdot\mathbf{E}\,dv$:- is the total (but instantaneous) ohmic power dissipated within the volume.
- $\int_{\text{vol}} \left(\frac{1}{2}\mathbf{D}\cdot\mathbf{E}\right) dv$ is the total energy stored in the electric field.
- $\int_{\text{vol}} \left(\frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right) dv$ is the total stored energy in the magnetic field.
- $\frac{d}{dt}\int_{\text{vol}} \left(\frac{1}{2}\mathbf{D}\cdot\mathbf{E}\right) dv$ and $\frac{d}{dt}\int_{\text{vol}} \left(\frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right) dv$ give the time rates of increase of energy stored within the volume.

- The sum therefore be the **total power flowing** into this volume, and so the total power flowing out of the volume is

$$\oint_{\text{area}} (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} \quad \text{W}$$

The integral is over the closed surface surrounding the volume.

- The cross product E × H is known as the **Poynting vector**, S,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \quad \text{W/m}^2$$

Instantaneous power density

The Poynting vector and Power

- Because S is given by the cross product of E and H, the **direction of power flow** at any point is normal to both the E and H vectors.
  - This certainly **agrees** with our experience with the uniform plane wave, for propagation in the $+z$ direction was associated with an E_x and H_y component,

$$E_x \mathbf{a}_x \times H_y \mathbf{a}_y = S_z \mathbf{a}_z$$

- In a **perfect dielectric**, the E and H field amplitudes are given by

$$E_x = E_{x0} \cos(\omega t - \beta z) \quad \& \quad H_y = \frac{E_{x0}}{\eta} \cos(\omega t - \beta z)$$

- where η is **real**.
- The **power density amplitude** is therefore

$$S_z = \frac{E_{x0}^2}{\eta} \cos^2(\omega t - \beta z)$$

The Poynting vector and Power

- In the case of a **lossy dielectric**, E_x and H_y are not in time phase. We have

$$E_x = E_{x0} e^{-\alpha z} \cos(\omega t - \beta z)$$

- Let $\eta = |\eta| \angle \theta_\eta$, then we may write the magnetic field intensity as

$$H_y = \frac{E_{x0}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)$$

- Thus

$$S_z = E_x H_y = \frac{E_{x0}^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta)$$

- The **time-average power density**, S_z, is the quantity that will ultimately be measured.
- To find this, we integrate **over one cycle** and divide by the period $T = 1/f$. Additionally, the identity $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$ is applied to the integrand, and we obtain:

$$\langle S_z \rangle = \frac{1}{T} \int_0^T \frac{1}{2} \frac{E_{x0}^2}{|\eta|} e^{-2\alpha z} \left[\cos(2\omega t - 2\beta z - \theta_\eta) + \cos \theta_\eta\right]$$

The Poynting vector and Power

- The **second-harmonic component** of the integrand integrates to zero, leaving only the contribution from the **dc component**. The result is

$$\langle S_z \rangle = \frac{1}{2}\frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}\cos\theta_\eta$$

- Note that the power density **attenuates** as $e^{-2\alpha z}$, whereas E_x and H_y fall off as $e^{-\alpha z}$.
- We may observe that the preceding expression can be obtained very easily by using the **phasor forms** of the electric and magnetic fields. In vector form, this is

$$\langle \mathbf{S} \rangle = \frac{1}{2}Re(\mathbf{E}_s \times \mathbf{H}_s^*) \qquad \mathrm{W/m^2}$$

- In the present case $\mathbf{E}_s = E_{x0}e^{-\alpha z}e^{-j\beta z}\mathbf{a}_x$

and $$\mathbf{H}_s^* = \frac{E_{x0}}{\eta^*}e^{-\alpha z}e^{+j\beta z}\mathbf{a}_y = \frac{E_{x0}}{|\eta|}e^{-\alpha z}e^{j\theta_\eta}e^{+j\beta z}\mathbf{a}_y$$

- where E_{x0} has been assumed real.

Propagation in good conductor

- **Good conductor** has a high conductivity, large conduction current and negligible displacement current.
- All the time varying fields are **attenuate very quickly** within the good conductor
- For good conductor the **loss tangent** $\tan\theta = \frac{\sigma}{\omega\epsilon'} \gg 1$
- The propagation constant is

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}}$$

- For good conductor the propagation constant is

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{j\frac{\sigma}{\omega\epsilon'}} = j\sqrt{-j\omega\mu\sigma}$$

$$-j = 1\angle - 90° \qquad \sqrt{1\angle - 90°} = 1\angle - 45° = \frac{1}{\sqrt{2}}(1 - j)$$

- The propagation constant is

$$jk = j(1 - j)\sqrt{\frac{\omega\mu\sigma}{2}} = (1 + j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta$$

$$\alpha = \beta = \sqrt{\pi f\mu\sigma}$$

Propagation in good conductor

- The attenuation and phase constant are equal and equal to

$$\alpha = \beta = \sqrt{\pi f \mu \sigma}$$

- The electric field is

$$E_x = E_{x0} e^{-z\sqrt{\pi f \mu \sigma}} \cos(\omega t - z\sqrt{\pi f \mu \sigma})$$

This is the electric field **outside** the conductor

- Let $z > 0$ good conductor and $z < 0$ perfect dielectric so at $z = 0$ the field is

$$E_x = E_{x0} \cos \omega t$$

- The conduction current density $\mathbf{J} = \sigma \mathbf{E}$
- The **conduction current density** at any point inside the conductor is

$$J_{\mathrm{x}} = \sigma E_x = \sigma E_{x0} e^{-z\sqrt{\pi f \mu \sigma}} \cos(\omega t - z\sqrt{\pi f \mu \sigma})$$

The electric field and current density are **decreases exponentially** inside the conductor

Propagation in good conductor

- The distance at which the field decrease by e^{-1} is

$$z = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

  - this distance is denoted by δ and is termed the **depth of penetration**, or the **skin depth**

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} = \frac{1}{\beta}$$

Electromagnetic wave **can not travel** within the good conductor.

- The constant are

$$\alpha = \beta = \frac{1}{\delta} = \sqrt{\pi f \mu \sigma}, \qquad \beta = \frac{2\pi}{\lambda}, \qquad \lambda = 2\pi\delta, \qquad v_p = \frac{\omega}{\beta} = \omega\delta$$

- The **intrinsic wave** impedance is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon'}} \quad \text{since } \sigma \gg \omega\epsilon' \Rightarrow \eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \frac{\sqrt{2}\angle 45^\circ}{\sigma\delta} = \frac{1}{\sigma\delta} + j\frac{1}{\sigma\delta}$$

Propagation in good conductor

- The field components are

$$E_x = E_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)$$

$$H_y = \frac{\sigma\delta E_{x0}}{\sqrt{2}} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right)$$

The maximum amplitude of the magnetic field occurs a 1/8 of cycle later than the maximum of the electric field at every point.

- **Time-average Poynting vector**

$$\langle S_z \rangle = \frac{1}{2}\frac{\sigma\delta E_{x0}^2}{\sqrt{2}} e^{-2z/\delta} \cos\left(\frac{\pi}{4}\right) = \frac{1}{4}\sigma\delta E_{x0}^2 e^{-2z/\delta}$$

- At the skin depth distance the power density has an $\boldsymbol{e^{-2} = 0.135}$ of its value at the surface.

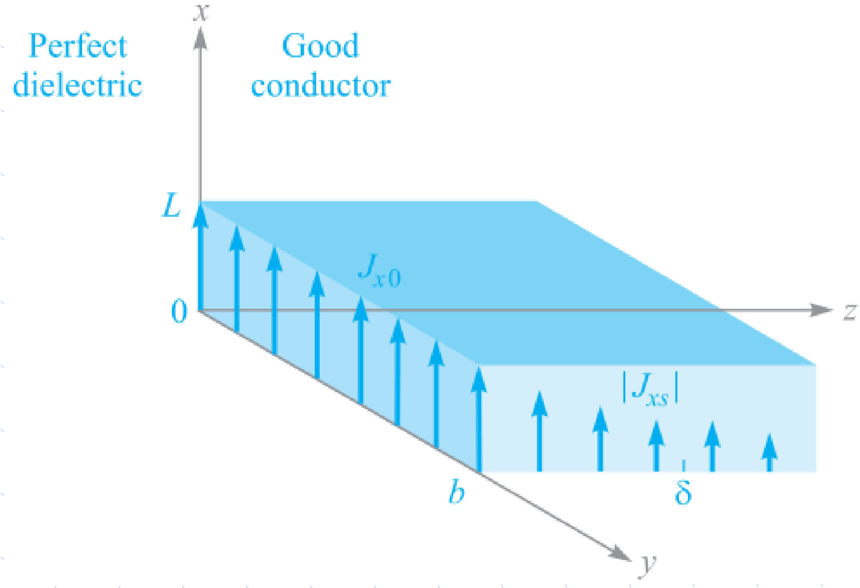
Propagation in good conductor

- The total average power loss in a width $0 < y < b$ and length $0 < x < L$ in the direction of the current, as shown in Figure, is obtained by finding the power **crossing the conductor surface** within this area,

$$P_L = \int_{\text{area}} \langle S_z \rangle da = \int_0^b \int_0^L \frac{1}{4} \sigma \delta E_{x0}^2 e^{-2z/\delta} \Big|_{z=0} dx\, dy = \frac{1}{4} \sigma \delta b L E_{x0}^2$$

- In terms of the current density J_{x0} at the surface,

$$J_{x0} = \sigma E_{x0}$$



- we have

$$P_L = \frac{1}{4\sigma} \delta b L J_{x0}^2$$

Propagation in good conductor

- The **power loss** would result if the total current in a width *b* were distributed uniformly in one skin depth.
- To find the **total current**, we integrate the current density over the infinite depth of the conductor,

$$I = \int_0^{\infty} \int_0^{b} J_x \; dy \; dz$$

where

$$J_x = J_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)$$

or in complex exponential notation to simplify the integration,

$$J_x = J_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)$$
$$= J_{x0} e^{-(1+j)z/\delta}$$

Propagation in good conductor

- Therefore,

$$I = \int_0^\infty \int_0^b J_{x0} e^{-(1+j)z/\delta}\ dy\ dz$$

$$= J_{x0} b e^{-(1+j)z/\delta} \frac{-\delta}{1+j}\Big|_0^\infty$$

$$= \frac{J_{x0} b\delta}{1+j}$$

- And

$$I = \frac{J_{x0} b\delta}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

- If the current is **uniformly distributed** with uniform density J'within the cross section $0 < y < b$ and $0 < z < \delta$

$$J' = \frac{I}{b\delta} = \frac{J_{x0}}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

- The **ohmic power loss per unit volume** is $J \cdot E$, and thus the total instantaneous power dissipated in the volume under consideration is

$$P_{Li}(t) = \frac{1}{\sigma}(J')^2 bL\delta = \frac{J_{x0}^2}{2\sigma} bL\delta \cos^2\left(\omega t - \frac{\pi}{4}\right)$$

Propagation in good conductor

- The time-average power loss is easily obtained, since the average value of the **cosine-squared factor** is **one-half**,

$$P_L = \frac{1}{4\sigma} J_{x0}^2 bL\delta$$

- This is the **same as above so**, the average power loss in conductor with skin effect present may be calculated by assuming the total current is uniformly in **one skin depth** .
- The resistance at a high frequency where there is a well-developed skin effect is therefore found by considering a slab of width equal to the circumference $2\pi a$ and thickness δ. Hence

$$R = \frac{L}{\sigma S} = \frac{L}{2\pi a\sigma\delta}$$

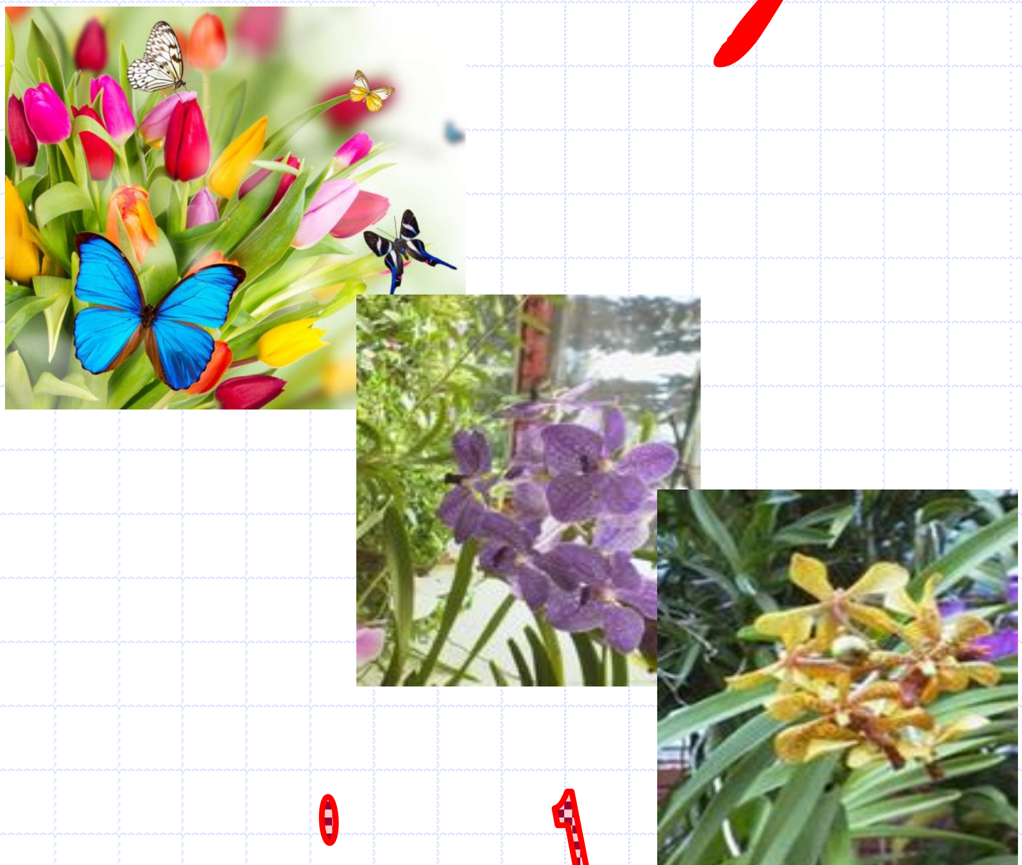
A round copper wire of 1 mm radius and 1 km length has a resistance at direct current of

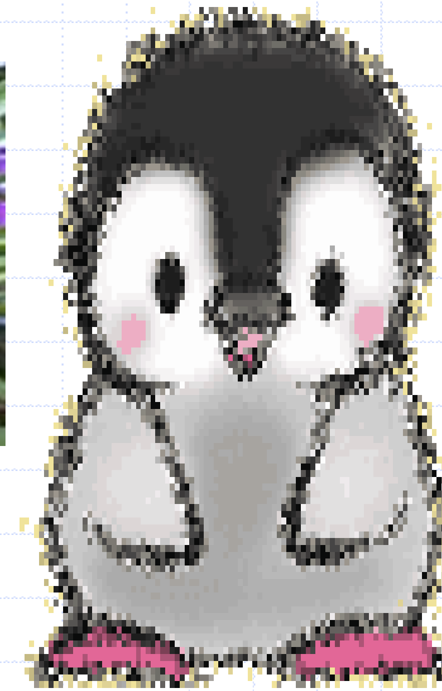
$$R_{dc} = \frac{10^3}{\pi 10^{-6}(5.8 \times 10^7)} = 5.48\ \ \Omega$$

At 1 MHz, the skin depth is 0.066 mm. Thus $\delta \ll a$, and the resistance at 1 MHz is

$$R = \frac{10^3}{2\pi 10^{-3}(5.8 \times 10^7)(0.066 \times 10^{-3})} = 41.5\ \ \Omega$$

Thank you





Have a nice day.