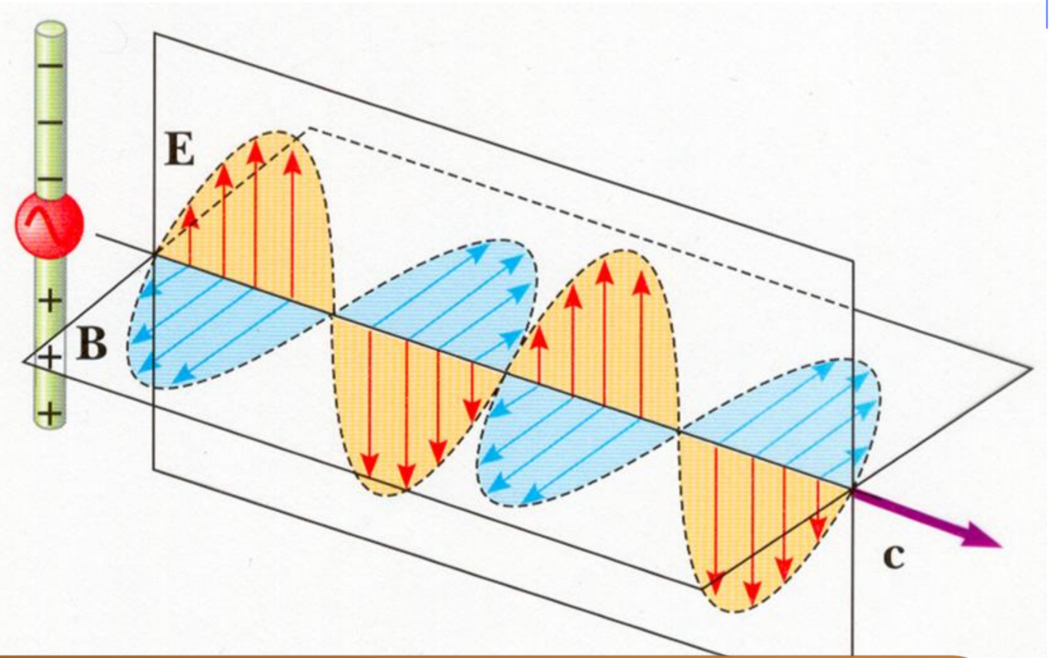
Lecture 10

Plane Wave Reflection

Electromagnetic Field Theory



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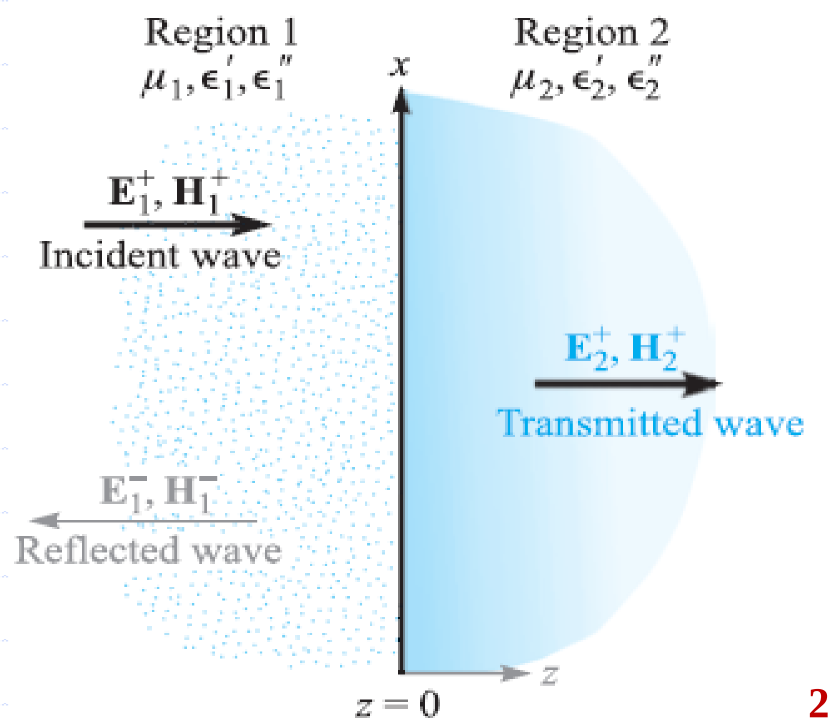
REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

- We consider the phenomenon of **reflection** which occurs when a **uniform plane** wave is incident on the **boundary** between regions composed of two different materials.
- The treatment is specialized to the case of ***normal incidence***—in which the wave propagation direction is perpendicular to the boundary.



- Define **region 1 (ϵ_1, μ_1)** as the half-space for which $z < 0$; **region 2 (ϵ_2, μ_2)** is the half-space for which $z > 0$.

REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

- Let a **wave in region 1**, traveling in the $+z$ direction toward the boundary surface at $z = 0$, and linearly polarized along x

$$\mathcal{E}_{x1}^{+}(z,t) = E_{x10}^{+} e^{-\alpha_1 z} \cos(\omega t - \beta_1 z)$$

⇒ incident wave

- In phasor form

$jk = \alpha + j\beta$

$$E_{xs1}^{+}(z) = E_{x10}^{+} e^{-jk_1 z}$$

- E_{x10}^{+} is real. The **subscript 1** identifies the region, and the **superscript +** indicates a positively traveling wave.
- Associated with $E_{xs1}^{+}(z)$ is a **magnetic field** in the y direction,

$$H_{ys1}^{+}(z) = \frac{1}{\eta_1} E_{x10}^{+} e^{-jk_1 z}$$

where k_1 and η_1 are complex unless ϵ_1'' (or σ_1) is zero.

- Since the direction of propagation of the incident wave is perpendicular to the boundary plane ⇒ *normal incidence*.

REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

- The **energy may be transmitted** across the boundary surface at $z = 0$ into region 2 by providing a wave moving in the $+z$ direction in that medium.
- The phasor electric and magnetic fields for this wave are

$$E_{xs2}^{+}(z) = E_{x20}^{+} e^{-jk_2 z}$$

$$H_{ys2}^{+}(z) = \frac{1}{\eta_2} E_{x20}^{+} e^{-jk_2 z}$$

- This wave, which moves away from the boundary surface into region 2, is called the **transmitted wave**.
- Note the use of the different propagation constant k_2 and intrinsic impedance η_2.
- Now we must satisfy the boundary conditions at $z = 0$ with these assumed fields.

REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

$$E_{xs1}^{+}(z) = E_{x10}^{+} e^{-jk_1 z} \quad \text{and} \quad E_{xs2}^{+}(z) = E_{x20}^{+} e^{-jk_2 z}$$

$$H_{ys1}^{+}(z) = \frac{1}{\eta_1} E_{x10}^{+} e^{-jk_1 z} \quad \text{and} \quad H_{ys2}^{+}(z) = \frac{1}{\eta_2} E_{x20}^{+} e^{-jk_2 z}$$

- With **E** polarized along x, the field is **tangent** to the interface, and therefore the **E** fields in regions l and 2 must be equal at $z = 0$.
  - Setting $z = 0$ would require that $E_{x10}^{+} = E_{x20}^{+}$.
- **H**, being y-directed, is also a tangential field, and must be continuous across the boundary (**no current sheets are present in real media**).
- Let $z = 0$, we find that we must have

$$\frac{1}{\eta_1} E_{x10}^{+} = \frac{1}{\eta_2} E_{x20}^{+}$$

- Since $E_{x10}^{+} = E_{x20}^{+} \quad \Rightarrow \quad \eta_1 = \eta_2$.
  - But this is a **very special condition** that does not fit the facts in general, and we are therefore **unable to satisfy** the boundary conditions with only an incident and a transmitted wave.

REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

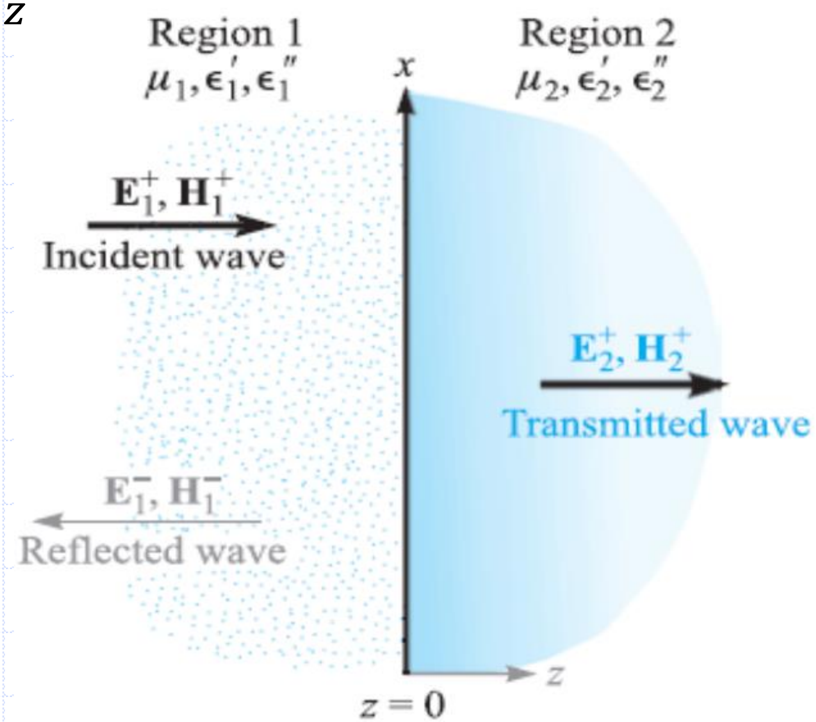
- We require a wave traveling away from the boundary in region 1, as shown in figure; this is the **reflected wave**,

$$E_{xs1}^{-}(z) = E_{x10}^{-} e^{jk_1 z}$$

$$H_{ys1}^{-}(z) = -\frac{1}{\eta_1} E_{x10}^{-} e^{jk_1 z}$$

  - E_{x10}^{-} may be a complex quantity.

- Because this field is traveling in the $-z$ direction, $E_{xs1}^{-} = -\eta_1 H_{ys1}^{-}$ for the Poynting vector shows that $\mathbf{E}_1^{-} \times \mathbf{H}_1^{-}$ must be in the $-\boldsymbol{a}_z$ direction.



- B. Cs. are now **easily satisfied**, and in the process the amplitudes of the transmitted and reflected waves may be found in terms of E_{x10}^{+}.

REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

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- The total electric field intensity is continuous at $z = 0$,

$$E_{xs1} = E_{xs2} \qquad (z = 0)$$

or

$$E^+_{xs1} + E^-_{xs1} = E^+_{xs2} \qquad (z = 0)$$

- Therefore

$$E^+_{x10} + E^-_{x10} = E^+_{x20} \qquad (1)$$

- Furthermore

$$H_{ys1} = H_{ys2} \qquad (z = 0)$$

or

$$H^+_{ys1} + H^-_{ys1} = H^+_{ys2} \qquad (z = 0)$$

- Therefore

$$\frac{E^+_{x10}}{\eta_1} - \frac{E^-_{x10}}{\eta_1} = \frac{E^+_{x20}}{\eta_2} \qquad (2)$$

Solving (2) for E^+_{x20} and substituting into (1), we find

$$E^+_{x10} + E^-_{x10} = \frac{\eta_2}{\eta_1} E^+_{x10} - \frac{\eta_2}{\eta_1} E^-_{x10} \quad \Rightarrow \quad E^-_{x10} = E^+_{x10} \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

- **Reflection coefficient**, Γ , is defined as the ratio of the amplitudes of the reflected and incident electric fields:

$$\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma| e^{j\phi} \quad (3)$$

It is evident that as η_1 or η_2 may be complex, Γ will also be complex, and so we include a reflective phase shift, φ.

- The relative amplitude of the transmitted electric field intensity is found by combining (3) and (1) $E_{x10}^{+} + E_{x10}^{-} = E_{x20}^{+}$ to yield the **transmission coefficient**, τ,

$$\tau = \frac{E_{x20}^{+}}{E_{x10}^{+}} = \frac{2\eta_2}{\eta_2 + \eta_1} = 1 + \Gamma = |\tau| e^{j\phi_i}$$

Special Case I

- Let region 1 be a **perfect dielectric** and region 2 be a **perfect conductor**. Then we apply $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2' - j\epsilon_2''}}$, with $\epsilon_2'' = \sigma_2/\omega$, obtaining

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2'}} = 0$$

- in which zero is obtained since $\sigma_2 \to \infty$. Therefore, from $\tau = \frac{E_{x20}^+}{E_{x10}^+} = \frac{2\eta_2}{\eta_2+\eta_1} \quad \Rightarrow \quad E_{x20}^+ = 0$
- No time-varying fields can exist in the perfect conductor.
  - An alternate way of looking at this is to note that the **skin depth is zero**. ($\delta = \frac{1}{\sqrt{\pi f \mu_2 \sigma_2}}$)
- Because $\eta_2 = 0$, $\Gamma = \frac{E_{x10}^-}{E_{x10}^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ shows that

$$\Gamma = -1 \qquad \text{and} \qquad E_{x10}^+ = -E_{x10}^-$$

Special Case I

- The incident and reflected fields are of equal amplitude, and so all the **incident energy is reflected** by the perfect conductor.
  - Moreover, the reflected field is **shifted in phase by 180°** relative to the incident field.
- The **total E** field in region 1 is

$$E_{xs1} = E_{xs1}^{+} + E_{xs1}^{-} = E_{x10}^{+}e^{-j\beta_1 z} - E_{x10}^{+}e^{j\beta_1 z}$$

where $jk_1 = 0 + j\beta_1$ in the perfect dielectric.

- These terms may be combined and simplified,

$$E_{xs1} = \left(e^{-j\beta_1 z} - e^{j\beta_1 z}\right)E_{x10}^{+} = -j2\sin(\beta_1 z)\,E_{x10}^{+}$$

- Multiplying by $e^{j\omega t}$ and taking the real part, we obtain the **real instantaneous form**:

$$\mathcal{E}_{x1}(z,t) = 2E_{x10}^{+}\sin(\beta_1 z)\sin(\omega t)$$

- We recognize this total field in region 1 as a ***standing wave***, obtained by combining two waves of equal amplitude traveling in opposite directions.

Standing Wave

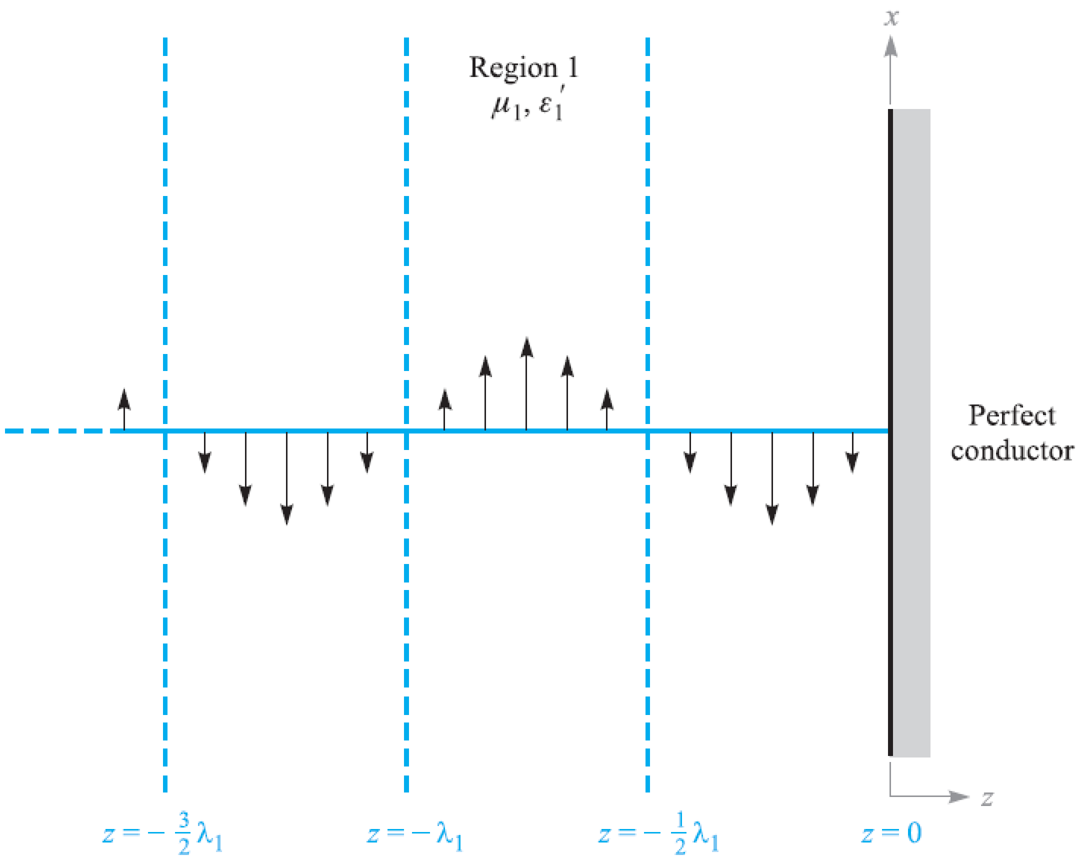
- Compare $\mathcal{E}_{x1}(z,t) = 2E_{x10}^{+}\sin(\beta_1 z)\sin(\omega t)$ to that of the **incident wave,**

$$\mathcal{E}_{x1}(z,t) = E_{x10}^{+}\cos(\omega t - \beta_1 z)$$

  - Here we see the term $\omega t - \beta_1 z$ or $\omega(t - z/\nu_{p1})$, which characterizes **a wave traveling** in the $+z$ direction at a velocity $\nu_{p1} = \omega/\beta_1$.
- In $\mathcal{E}_{x1}(z,t) = 2E_{x10}^{+}\sin(\beta_1 z)\sin(\omega t)$, however, the factors involving time and distance are separate trigonometric terms.
  - Whenever $\omega t = m\pi$, $\mathcal{E}_{x1}$ is **zero** at all positions.
- **Spatial nulls** in the standing wave pattern occur for all times wherever $\beta_1 z = m\pi, m = (0, \pm 1, \pm 2, \ldots)$.
- In such cases, $\beta_1 z = \frac{2\pi}{\lambda_1} z = m\pi$
- and the null locations occur at $z = m\frac{\lambda_1}{2}$

Standing Wave

- Thus $E_{x1} = 0$ at the boundary $z = 0$ and at every half-wavelength from the boundary in region 1, $z < 0$, as illustrated in Figure 12.2.



The instantaneous values of the total field E_{x1} are shown at $t = \pi/2$. $E_{x1} = 0$ for all time at multiples of one half-wavelength from the conducting surface.

Standing Wave

- Because $E_{xs1}^{+} = \eta_1 H_{ys1}^{+}$ and $E_{xs1}^{-} = -\eta_1 H_{ys1}^{-}$, the magnetic field is

$$H_{ys1} = \frac{E_{x10}^{+}}{\eta_1}\left(e^{-j\beta_1 z} + e^{j\beta_1 z}\right) = 2\frac{E_{x10}^{+}}{\eta_1}\cos(\beta_1 z)$$

In phasor form

$$H_{y1} = 2\frac{E_{x10}^{+}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)$$

- This is also a **standing wave**, but it shows a maximum amplitude at the positions where $E_{x1} = 0$. It is also 90° out of time phase with E_{x1} everywhere.
  - As a result, the **average power** as determined through the **Poynting vector** $\langle S \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{E}_s \times \mathbf{H}_s^*)$ is zero in the forward and backward directions.

Special Case II

- Let us now consider perfect dielectrics in both regions 1 and 2;
  - η_1 and η_2 are both real positive quantities and $\alpha_1 = \alpha_2 = 0$.
- Equation

$$\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

- enables us to calculate the reflection coefficient and find E_{x1}^{-} in terms of the incident field E_{x1}^{+}.
- Knowing E_{x1}^{+} and E_{x1}^{-}, we then find H_{y1}^{+} and H_{y1}^{-}.
- In region 2, E_{x2}^{+} is found from

$$\tau = \frac{E_{x20}^{+}}{E_{x10}^{+}} = \frac{2\eta_2}{\eta_2 + \eta_1} = 1 + \Gamma$$

- and this then determines H_{y2}^{+}.

Power Density

- For the **incident power density**, we have

$$\langle S_{1i}\rangle = \frac{1}{2}\mathrm{Re}\{E_{xs1}^{+}H_{ys1}^{+*}\} = \frac{1}{2}\mathrm{Re}\left\{E_{x10}^{+}\frac{1}{\eta_1^*}E_{x10}^{+*}\right\} = \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_1^*}\right\}|E_{x10}^{+}|^2$$

- The **reflected power density** is then

$$\langle S_{1r}\rangle = -\frac{1}{2}\mathrm{Re}\{E_{xs1}^{-}H_{ys1}^{-*}\} = \frac{1}{2}\mathrm{Re}\left\{\Gamma E_{x10}^{+}\frac{1}{\eta_1^*}\Gamma^* E_{x10}^{+*}\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_1^*}\right\}|E_{x10}^{+}|^2|\Gamma|^2$$

- We thus find the general relation between the reflected and incident power:

$$\langle S_{1r}\rangle = |\Gamma|^2\langle S_{1i}\rangle$$

Power Density

- The transmitted power density:

$$\langle S_2 \rangle = \frac{1}{2}\mathrm{Re}\{E_{xs2}^{+}H_{ys2}^{+*}\} = \frac{1}{2}\mathrm{Re}\left\{\tau E_{x10}^{+}\frac{1}{\eta_2^*}\tau^* E_{x10}^{+*}\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_2^*}\right\}|E_{x10}^{+}|^2|\tau|^2$$

- Taking the advantage of energy conservation by noting that whatever power is not reflected must be transmitted.

$$\langle S_2 \rangle = (1 - |\Gamma|^2)\langle S_{1i} \rangle$$

STANDING WAVE RATIO

- When $|\Gamma| < 1$, some energy is transmitted into the second region and some is reflected.
  - Region 1 therefore supports a field that is composed of both a traveling-wave and a standing-wave.
- **Medium 1** is assumed to be a **perfect dielectric** ($\alpha_1 = 0$), but **region 2** may be **any material**.
- The **total electric field** phasor in region 1 will be

$$E_{x1T} = E_{x1}^{+} + E_{x1}^{-} = E_{x10}^{+} e^{-j\beta_1 z} + \Gamma E_{x10}^{+} e^{j\beta_1 z}$$
$$= \left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z + \phi)}\right) E_{x10}^{+}$$

- where the reflection coefficient: $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma| e^{j\phi}$
- We allow for the possibility of a **complex reflection coefficient** by including its phase, φ.
  - Although $\boldsymbol{\eta_1}$ **is real** and positive for a lossless medium, $\boldsymbol{\eta_2}$ will in general be **complex**.
  - if **region 2 is a perfect conductor**, η_2 is zero, and so φ is equal to π;
  - if $\boldsymbol{\eta_2}$ **is real and less than** $\boldsymbol{\eta_1}$, φ is also equal to π; and if η_2 is real and greater than η_1, φ is zero.

STANDING WAVE RATIO

We have a maximum when each term in the larger parentheses $E_{x1T} = \left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z+\phi)}\right) E_{x10}^{+}$ has the same phase angle; so, for E_{x10}^{+} positive and real,

$$|E_{x1T}|_{max} = (1 + |\Gamma|) E_{x10}^{+}$$

- and this occurs where

$$-\beta_1 z = \beta_1 z + \phi + 2m\pi \qquad (m = 0, \pm 1, \pm 2, \dots)$$

- Therefore

$$z_{\max} = -\frac{1}{2\beta_1}(\phi + 2m\pi)$$

- Note that an **electric field maximum** is located at the boundary plane $(z = 0)$ if $\phi = 0$; moreover, $\phi = 0$ when Γ is real and positive.
- This occurs for real η_1 and η_2 when $\eta_2 > \eta_1$.
  - Thus there is a **field maximum at the boundary surface** $\eta_2 > \eta_1$ and both are real.
- With $\boldsymbol{\phi = 0}$, maxima also occur at $z_{\max} = -m\pi/\beta_1 = -m\lambda_1/2$.
- For **perfect conductor** $\boldsymbol{\phi = \pi}$, and these maxima are found at $z_{\max} = -\pi/(2\beta_1), -3\pi/(2\beta_1)$, or $z_{\max} = -\lambda_1/4, -3\lambda_1/4$, and so forth.

STANDING WAVE RATIO

- The **minima** must occur where the phase angles of the two terms in the larger parentheses in $E_{x1T} = \left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z + \phi)}\right) E_{x10}^{+}$ differ by 180◦, thus

$$|E_{x1T}|_{\min} = (1 - |\Gamma|) E_{x10}^{+}$$

- and this occurs where

$$-\beta_1 z = \beta_1 z + \phi + \pi + 2m\pi \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

- or

$$z_{\min} = -\frac{1}{2\beta_1}(\phi + (2m+1)\pi)$$

- The minima are separated by multiples of $\lambda/2$ (as are the maxima), and for the **perfect conductor** the first minimum occurs when $-\beta_1 z = 0$, or at the conducting surface.
- In general, **an electric field minimum** is found at $z = 0$ whenever $\phi = \pi$; this occurs if $\eta_2 < \eta_1$ and both are real.

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STANDING WAVE RATIO

- Standing wave ratio: is the ratio of the maximum to minimum amplitudes:

$$s = \frac{|E_{x1T}|_{\max}}{|E_{x1T}|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

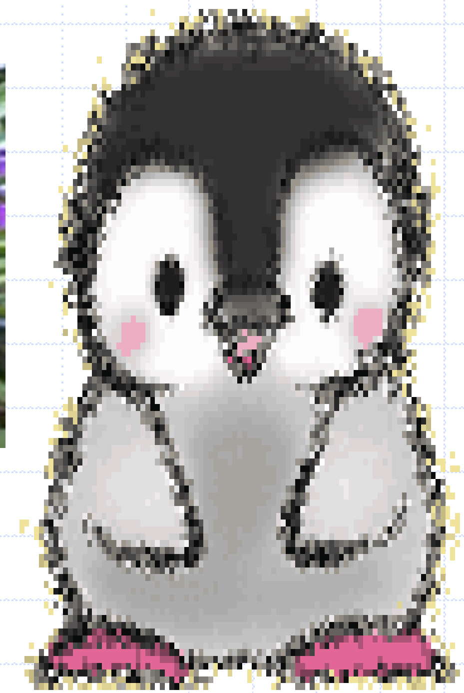
Because $|\Gamma| < 1$, s is always positive and greater than or equal to unity.

- If $|\Gamma| = 1$, the reflected and incident amplitudes are equal, all the incident energy is reflected, and s is infinite.
  - Planes separated by multiples of $\lambda_1/2$ can be found on which E_{x1} is zero at all times. Midway between these planes, E_{x1} has a maximum amplitude twice that of the incident wave.
- If $\eta_2 = \eta_1$, then $\Gamma = 0$, **no energy is reflected**, and $s = 1$; the maximum and minimum amplitudes are equal.
- If one-half the incident power is reflected, $|\Gamma|^2 = 0.5$, $|\Gamma| = 0.707$, and $s = 5.83$.

Thank you



Have a nice day.



STANDING WAVE RATIO

- Further insights can be obtained by working with Eq. (19) and rewriting it in real instantaneous form.
- We find the total field in region 1 to be

$$\mathcal{E}_{x1T}(z,t) = \underbrace{(1-|\Gamma|)E_{x10}^{+}\cos(\omega t-\beta_1 z)}$$

- The field expressed in Eq. (26) is the sum of a traveling wave of

$$\mathcal{E}_{x1T}(z,t) = \underbrace{(1-|\Gamma|)E_{x10}^{+}\cos(\omega t-\beta_1 z)}_{\text{traveling wave}} + \underbrace{2|\Gamma|E_{x10}^{+}\cos(\beta_1 z+\phi/2)\cos(\omega t+\phi/2)}_{\text{standing wave}}$$

having amplitude $2|\Gamma|E_{x10}^{+}$.
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(26) add directly to give $(1+|\Gamma|)E_{x10}^{+}$. The minimum amplitude is found where the standing wave achieves a null, leaving only the traveling wave amplitude of $(1-|\Gamma|)E_{x10}^{+}$.